A Collection of Dice Problems

with solutions and useful appendices

(a work continually in progress)

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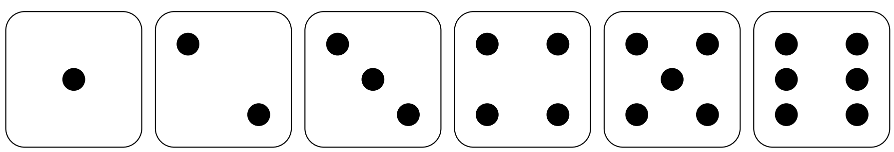

Matthew M. Conroy

doctormatt “at” madandmoonly dot com

www.matthewconroy.com

Thanks

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Chapter 1

Introduction and Notes

This is a (slowly) growing collection of dice-related mathematical problems, with accompanying solutions. Some are simple exercises suitable for beginners, while others require more sophisticated techniques.

Many dice problems have an advantage over some other problems of probability in that they can be investigated experimentally. This gives these types of problems a certain helpful down-to-earth feel.

Please feel free to comment, criticize, or contribute additional problems.

1.0.1 What are dice?

In the real world, dice (the plural of *die*) are polyhedra made of plastic, wood, ivory, or other hard material. Each face of the die is numbered, or marked in some way, so that when the die is cast onto a smooth, flat surface and allowed to come to rest, a particular number is specified.

Mathematically, we can consider a die to be a random variable that takes on only finitely many distinct values. Usually, these values will constitute a set of positive integers $1, 2, ..., n$; in such cases, we will refer to the die as n-sided.

1.0.2 Terminology

A *fair* die is one for which each face appears with equal likelihood. A non-fair die is called *fixed*. The phrase *standard die* will refer to a fair, six-sided die, whose faces are numbered one through six. If not otherwise specified, the term *die* will refer to a standard die.

Chapter 2

Problems

2.1 Standard Dice

1. On average, how many times must a 6-sided die be rolled until a 6 turns up?
2. On average, how many times must a 6-sided die be rolled until a 6 turns up twice in a row?
3. On average, how many times must a 6-sided die be rolled until the sequence 65 appears (i.e., a 6 followed by a 5)?
4. On average, how many times must a 6-sided die be rolled until there are two rolls in a row that differ by 1 (such as a 2 followed by a 1 or 3, or a 6 followed by a 5)? What if we roll until there are two rolls in a row that differ by no more than 1 (so we stop at a repeated roll, too)?
5. We roll a 6-sided die n times. What is the probability that all faces have appeared?
6. Person A rolls n dice and person B rolls m dice. What is the probability that they have a common face showing (e.g., person A rolled a 2 and person B also rolled a 2, among all their dice)?
7. On average, how many times must a 6-sided die be rolled until all sides appear at least once? What about for an n-sided die?
8. On average, how many times must a pair of 6-sided dice be rolled until all sides appear at least once?
9. Suppose we roll n dice. What is the expected number of distinct faces that appear?
10. Suppose we roll n dice and keep the highest one. What is the distribution of values?
11. Suppose we can roll a 6-sided die up to n times. At any point we can stop, and that roll becomes our "score". Our goal is to get the highest possible score, on average. How should we decide when to stop?
12. How many dice must be rolled to have at least a 95% chance of rolling a six?
13. How many dice must be rolled to have at least a 95% chance of rolling a one and a two? What about a one, a two, and a three? What about a one, a two, a three, a four, a five and a six?
14. How many dice should be rolled to maximize the probability of rolling exactly one six? two sixes? n sixes?

15. Suppose we roll a fair die 100 times. What is the probability of a run of at least 10 sixes?

16. Suppose we roll a fair die until some face has appeared twice. For instance, we might have a run of rolls 12545 or 636. How many rolls on average would we make? What if we roll until a face has appeared *three* times?

17. Suppose we roll a fair die 10 times. What is the probability that the sequence of rolls is non-decreasing (i.e., the next roll is never less than the current roll)?

18. Suppose a pair of dice are thrown, and then thrown again. What is the probability that the faces appearing on the second throw are the same as the first?

    What if three dice are used? Or six?

19. A single die is rolled until a run of six different faces appears. For example, one might roll the sequence 535463261536435344151612534 with only the last six rolls all distinct. What is the expected number of rolls?

20. What is the most probable: rolling at least one six with six dice, at least two sixes with twelve dice, or at least three sixes with eighteen dice? (This is an old problem, frequently connected with Isaac Newton.)

21. Suppose we roll n dice, remove all the dice that come up 1, and roll the rest again. If we repeat this process, eventually all the dice will be eliminated. How many rolls, on average, will we make? Show, for instance, that on average fewer than $O(\log n)$ throws occur.

22. Suppose we roll a die $6k$ times. What is the probability that each possible face comes up an equal number of times (i.e., k times)? Find an asymptotic expression for this probability in terms of k.

23. Call a "consecutive difference" the absolute value of the difference between two consecutive rolls of a die. For example, the sequence of rolls 14351 has the corresponding sequence of consecutive differences $3, 1, 2, 4$. What is the expected number of times we need to roll a die until all 6 consecutive differences have appeared?

2.2 Dice Sums

24. Show that the probability of rolling 14 is the same whether we throw 3 dice or 5 dice. Are there other examples of this phenomenon?

25. Suppose we roll n dice and sum the highest 3. What is the probability that the sum is 18?

26. Four fair, 6-sided dice are rolled. The highest three are summed. What is the distribution of the sum?

27. A fair, n-sided die is rolled until a roll of k or greater appears. All rolls are summed. What is the expected value of the sum?

28. A pair of dice is rolled repeatedly. What is the expected number of rolls until all eleven possible sums have appeared? What if three dice are rolled until all sixteen possible sums have appeared?

29. A die is rolled repeatedly and summed. What can you say about the expected number of rolls until the sum is greater than or equal to n?

30. A die is rolled repeatedly and summed. Show that the expected number of rolls until the sum is a multiple of n is n.

31. A fair, n-sided die is rolled and summed until the sum is at least n. What is the expected number of rolls?

32. A die is rolled once; call the result N. Then N dice are rolled once and summed. What is the distribution of the sum? What is the expected value of the sum? What is the most likely value?

    What the heck, take it one more step: roll a die; call the result N. Roll N dice once and sum them; call the result M. Roll M dice once and sum. What's the distribution of the sum, expected value, most likely value?

33. A die is rolled and summed repeatedly. What is the probability that the sum will ever be a given value x? What is the limit of this probability as $x \to \infty$?

34. A die is rolled once. Call the result N. Then, the die is rolled N times, and those rolls which are equal to or greater than N are summed (other rolls are not summed). What is the distribution of the resulting sum? What is the expected value of the sum?

35. Suppose n six-sided dice are rolled and summed. For each six that appears, we sum the six, and reroll that die and sum, and continue to reroll and sum until we roll something other than a six with that die. What is the expected value of the sum? What is the distribution of the sum?

36. A die is rolled until all sums from 1 to x are attainable from some subset of rolled faces. For example, if $x = 3$, then we might roll until a 1 and 2 are rolled, or until three 1s appear, or until two 1s and a 3. What is the expected number of rolls?

37. How long, on average, do we need to roll a die and sum the rolls until the sum is a perfect square $(1, 4, 9, 16, \dots)$?

38. How long, on average, do we need to roll a die and sum the rolls until the sum is prime? What if we roll until the sum is composite?

2.3 Non-Standard Dice

39. Show that the probability of rolling doubles with a non-fair ("fixed") die is greater than with a fair die.

40. Is it possible to have a non-fair six-sided die such that the probability of rolling $2, 3, 4, 5$, and 6 is the same whether we roll it once or twice (and sum)? What about for other numbers of sides?

41. Find a pair of 6-sided dice, labelled with positive integers differently from the standard dice, so that the sum probabilities are the same as for a pair of standard dice.

42. Is it possible to have two non-fair n-sided dice, with sides numbered 1 through n, with the property that their sum probabilities are the same as for two fair n-sided dice?

43. Is it possible to have two non-fair 6-sided dice, with sides numbered 1 through 6, with a uniform sum probability? What about n-sided dice?

44. Suppose that we renumber three fair 6-sided dice (A, B, C) as follows: $A = \{2, 2, 4, 4, 9, 9\}$,$B = \{1, 1, 6, 6, 8, 8\}$, and $C = \{3, 3, 5, 5, 7, 7\}$.

(a) Find the probability that die A beats die B; die B beats die C; die C beats die A.

(b) Discuss.

45. Find every six-sided die with sides numbered from the set $\{$1,2,3,4,5,6$\}$ such that rolling the die twice and summing the values yields all values between 2 and 12 (inclusive). For instance, the die numbered 1,2,4,5,6,6 is one such die. Consider the sum probabilities of these dice. Do any of them give sum probabilities that are "more uniform" than the sum probabilities for a standard die? What if we renumber two dice differently - can we get a uniform (or more uniform than standard) sum probability?

2.4 Games with Dice

46. Two players each roll a die. Player 1 rolls a fair m-sided die, while player 2 rolls a fair n sided die, with $m > n$. The winner is the one with the higher roll. What is the probability that Player 1 wins? What is the probability of a tie? If the players continue rolling in the case of a tie until they do not tie, which player has the higher probability of winning? If the tie means a win for Player 1 (or player 2), what is their probability of winning?

47. **Craps** The game of craps is perhaps the most famous of all dice games. The player begin by throwing two standard dice. If the sum of these dice is 7 or 11, the player wins. If the sum is 2,3 or 12, the player loses. Otherwise, the sum becomes the player's *point*. The player continues to roll until either the point comes up again, in which case the player wins, or the player throws 7, in which case they lose. The natural question is: what is a player's probability of winning?

48. **Non-Standard Craps** We can generalize the games of craps to allow dice with other than six sides. Suppose we use two (fair) n-sided dice. Then we can define a game analogous to craps in the following way. The player rolls two n-sided dice. If the sum of these dice is $n + 1$ or $2n - 1$, the player wins. If the sum of these dice is $2, 3$ or $2n$, then the player loses. Otherwise the sum becomes the player's point, and they win if they roll that sum again before rolling $n + 1$. We may again ask: what is the player's probability of winning?

49. **Yahtzee** There are many probability questions we may ask with regard to the game of Yahtzee. For starters, what is the probability of rolling, in a single roll,

    (a) Yahtzee
    (b) Four of a kind (but not Yahtzee)
    (c) Three of a kind (but not four of a kind or Yahtzee)
    (d) A full house
    (e) A long straight
    (f) A small straight

50. **More Yahtzee** What is the probability of getting Yahtzee, assuming that we are trying just to get Yahtzee, we make reasonable choices about which dice to re-roll, and we have three rolls? That is, assume we're in the situation where all we have left to get in a game of Yahtzee *is* Yahtzee, so that all other outcomes are irrelevant.

51. **Drop Dead** In the game of Drop Dead, the player starts by rolling five standard dice. If there are no 2's or 5's among the five dice, then the dice are summed and this is the player's score. If there are 2's or 5's, these dice become "dead" and the player gets no score. In either case, the player continues by rolling all non-dead dice, adding points onto the score, until all dice are dead.

    For example, the player might roll $\{1, 3, 3, 4, 6\}$ and score 17. Then they roll all the dice again and get $\{1, 1, 2, 3, 5\}$ which results in no points and two of the dice dying. Rolling the three remaining dice, they might get $\{2, 3, 6\}$ for again no score, and one more dead die. Rolling the remaining two they might get $\{4, 6\}$ which gives them 10 points, bringing the score to 27. They roll the two dice again, and get $\{2, 3\}$ which gives no points and another dead die. Rolling the remaining die, they might get $\{3\}$ which brings the score to 30. Rolling again, they get $\{5\}$ which brings this player's round to an end with 30 points.

    Some natural questions to ask are:

    (a) What is the expected value of a player's score?
    (b) What is the probability of getting a score of 0? 1? 20? etc.

52. **Threes** In the game of Threes, the player starts by rolling five standard dice. In the game, the threes count as zero, while the other faces count normally. The goal is to get as low a sum as possible. On each roll, at least one die must be kept, and any dice that are kept are added to the player's sum. The game lasts at most five rolls, and the score can be anywhere from 0 to 30.

    For example a game might go like this. On the first roll the player rolls

    $$2 - 3 - 3 - 4 - 6$$

    The player decides to keep the 3s, and so has a score of zero. The other three dice are rolled, and the result is

    $$1 - 5 - 5$$

    Here the player keeps the 1, so their score is 1, and re-rolls the other two dice. The result is

    $$1 - 2$$

    Here, the player decides to keep both dice, and their final score is 4.

    If a player plays optimally (i.e., using a strategy which minimizes the expected value of their score), what is the expected value of their score?

53. Suppose we play a game with a die where we roll and sum our rolls as long as we keep rolling larger values. For instance, we might roll a sequence like 1-3-4 and then roll a 2, so our sum would be 8. If we roll a 6 first, then we're through and our sum is 6. Three questions about this game:

    (a) What is the expected value of the sum?
    (b) What is the expected value of the number of rolls?
    (c) If the game is played with an n-sided die, what happens to the expected number of rolls as n approaches infinity?

54. Suppose we play a game with a die where we roll and sum our rolls. We can stop any time, and the sum is our score. However, if our sum is ever a multiple of 10, our score is zero, and our game is over. What strategy will yield the greatest expected score? What about the same game played with values other than 10?

55. Suppose we play a game with a die in which we use two rolls of the die to create a two digit number. The player rolls the die once and decides which of the two digits they want that roll to represent. Then, the player rolls a second time and this determines the other digit. For instance, the player might roll a 5, and decide this should be the "tens" digit, and then roll a 6, so their resulting number is 56.

    What strategy should be used to create the largest number on average? What about the three digit version of the game?

56. Suppose we roll a single die, repeatedly if we like, and sum. We can stop at any point, and the sum becomes our score; however, if we exceeed 10, our score is zero.

    What should our strategy be to maximize the expected value of our score? What is the expected score with this optimal strategy?

    What about limits besides 10?

Chapter 3

Discussion, Hints, and Solutions

3.1 Single Die Problems

1. *On average, how many times must a* 6*-sided die be rolled until a 6 turns up?*

   This problem is asking for the *expected* number of rolls until a 6 appears. Let X be the random variable representing the number of rolls until a 6 appears. Then the probability that $X = 1$ is $1/6$; the probability that $X = 2$ is $(5/6)(1/6) = 5/36$. In general, the probability that $X = k$ is

   $$\left(\frac{5}{6}\right)^{k-1}\frac{1}{6} \tag{3.1}$$

   since, in order for X to be k, there must be $k-1$ rolls which can be any of the numbers 1 through 5, and then a 6, which appears with probability $1/6$.

   We seek the expectation of X. This is defined to be

   $$E = \sum_{n=1}^{\infty} nP(X = n) \tag{3.2}$$

   where $P(X = n)$ is the probability that X takes on the value n. Thus,

   $$E = \sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}\frac{1}{6} = \frac{6}{5}\cdot\frac{1}{6}\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n} \tag{3.3}$$

   Using Equation B.6 from Appendix B, we can conclude that

   $$E = \frac{6}{5}\cdot\frac{1}{6}\frac{5/6}{(1-(5/6))^2} = 6. \tag{3.4}$$

   Thus, on average, it takes 6 throws of a die before a 6 appears.

   Here's another, quite different way to solve this problem. When rolling a die, there is a $1/6$ chance that a 6 will appear. If a 6 doesn't appear, then we're in essence starting over. That is to say, the number of times we expect to throw the die before a 6 shows up is the same as the number of additional times we expect to throw the die after throwing a non-6. So we have a $1/6$ chance of rolling a 6 (and stopping), and a $5/6$ chance of not rolling a six, after which the number of rolls we expect to throw is the same as when we started. We can formulate this as

$$E = \frac{1}{6} + \frac{5}{6}(E+1). \tag{3.5}$$

Solving for E, we find $E = 6$. Note that Equation 3.5 implicitly assumes that E is a finite number, which is something that, *a priori*, we do not necessarily know.

2. *On average, how many times must a* 6*-sided die be rolled until a 6 turns up twice in a row?*

   We can solve this using a recurrence relation on, E, the expected number of rolls.

   When we start rolling, we expect, on average 6 rolls until a 6 shows up. Once that happens, there is a 1/6 chance that we will roll once more, and a 5/6 chance that we will be, effectively, starting all over again, and so have as many additional expected rolls as when we started. As a result, we can say

   $$E = 6 + \frac{1}{6} \cdot 1 + \frac{5}{6}(E+1).$$

   Solving this, we find that $E = 42$.

3. *On average, how many times must a* 6*-sided die be rolled until the sequence 65 appears (i.e., a 6 followed by a 5)?*

   This appears to be quite similar to problem 2, but there is a difference. In problem 2, once we roll a 6, there are only two possibilities: either we roll a 6, or we start all over again.

   In this problem, once we roll a 6, there are three possibilities: (a) we roll a 5, (b), we roll a 6, or (c) we start all over again.

   We can again solve it using recursion, but we'll need two equations. Let E be the expected number of rolls until 65 and let E_6 be the expected number of rolls until 65 when we start with a rolled 6. Then:

   $$\begin{aligned} E_6 &= \frac{1}{6}(E_6+1) + \frac{4}{6}(E+1) + \frac{1}{6}(1) \\ E &= \frac{1}{6}(E_6+1) + \frac{5}{6}(E+1) \end{aligned}$$

   This gives us a system of two linear equations in two unknowns, which we can solve to find

   $$E = 36, E_6 = 30.$$

   So it takes fewer rolls on average to see a 6 followed by a 5 than it does to see a 6 followed by a 6.

4. *On average, how many times must a* 6*-sided die be rolled until there are two rolls in a row that differ by* 1 *(such as a* 2 *followed by a* 1 *or* 3*, or a* 6 *followed by a* 5*)? What if we roll until there are two rolls in a row that differ by no more than* 1 *(so we stop at a repeated roll, too)?*

   Let E be the expected number of rolls. Let E_i be the expected number of rolls after rolling an i (not following a roll of $i-1$ or $i+1$).

   Then we have

   $$E = 1 + \frac{1}{6}(E_1 + E_2 + E_3 + E_4 + E_5 + E_6).$$

   By symmetry, we know that $E_1 = E_6$, $E_2 = E_5$ and $E_3 = E_4$, so

   $$E = 1 + \frac{2}{6}(E_1 + E_2 + E_3).$$

We can express E_1 as

$$E_1 = 1 + \frac{2}{6}E_1 + \frac{1}{6}E_2 + \frac{2}{6}E_3$$

since there will definitely be an additional roll, there is a $\frac{1}{6}$ chance that this will be the last roll (i.e., we roll a 2) and the five other possibilities are equally likely.

Similarly,

$$E_2 = 1 + \frac{1}{6}E_1 + \frac{2}{6}E_2 + \frac{1}{6}E_3$$

and

$$E_3 = 1 + \frac{2}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3.$$

This gives us a system of three linear equations in three unknowns.

Solving, we find

$$E_1 = \frac{70}{17}, E_2 = \frac{58}{17}, \text{ and } E_3 = \frac{60}{17}$$

and so

$$E = \frac{239}{51} = 4.68627450980....$$

If we stop when we have a repeated roll, too, the situation is similar. Defining E, E_1, E_2, and E_3 as above, we have the system

$$\begin{aligned}
E &= 1 + \frac{2}{6}(E_1 + E_2 + E_3) \\
E_1 &= 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{2}{6}E_3 \\
E_2 &= 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 \\
E_3 &= 1 + \frac{2}{6}E_1 + \frac{1}{6}E_2.
\end{aligned}$$

Solving this, we find

$$E_1 = \frac{288}{115}, E_2 = \frac{246}{115}, \text{ and } E_3 = \frac{252}{115}.$$

and so

$$E = \frac{377}{115} = 3.278260869565....$$

5. *We roll a* 6*-sided die* n *times. What is the probability that all faces have appeared?*

   Let $P(n)$ stand for the probability that all faces have appeared in n rolls.

   To determine $P(n)$, we can use the **principle of inclusion-exclusion**.

   We wish to count the number of roll sequences that do not contain all faces.

   There are 6^n ways to roll a die n times.

   Of these, 5^n have no 1, 5^n have no 2, etc.

   Simply adding those will not yield what we seek, since there are roll sequences that contain no 1 and no 2 (for example), so we would be counting those twice.

   As a result, we take that sum and subtract all the roll sequences with both no 1 and no 2, or both no 1 and no 3, etc.

Again, we will not have quite what we wish, since we will have removed sequences that contain, say, both no 1 and no 2, *and* no 1 and no 3, twice.

Hence, we have to add back in the number of sequences that fail to have three faces.

We continue in this way, alternating subtracting and adding numbers of sequences, until we reach the final count: no sequence can fail to have all 6 faces.

All together, then, we find that the number of sequences that fail to have all 6 faces is

$$\binom{6}{1}5^n - \binom{6}{2}4^n + \binom{6}{3}3^n - \binom{6}{4}2^n + \binom{6}{5}1^n.$$

Hence, the probability of having all 6 faces appear in n rolls of the die is

$$\begin{aligned}
&1 - \binom{6}{1}\left(\frac{5}{6}\right)^n + \binom{6}{2}\left(\frac{4}{6}\right)^n - \binom{6}{3}\left(\frac{3}{6}\right)^n + \binom{6}{4}\left(\frac{2}{6}\right)^n - \binom{6}{5}\left(\frac{1}{6}\right)^n \\
&= 1 - 6\left(\frac{5}{6}\right)^n + 15\left(\frac{2}{3}\right)^n - 20\left(\frac{1}{2}\right)^n + 15\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{6}\right)^n \\
&= \frac{6^n - 6\cdot 5^n + 15\cdot 4^n - 20\cdot 3^n + 15\cdot 2^n - 6}{6^n}.
\end{aligned}$$

Here is a short table of values of this probability.

n	$P(n)$ exactly	$P(n)$ approximately
1	0	0
2	0	0
3	0	0
4	0	0
5	0	0
6	5/324	0.01543210
7	35/648	0.05401235
8	665/5832	0.11402606
9	245/1296	0.18904321
10	38045/139968	0.27181213
11	99715/279936	0.35620642
12	1654565/3779136	0.43781568
13	485485/944784	0.51385819
14	317181865/544195584	0.58284535
15	233718485/362797056	0.64421274
16	2279105465/3265173504	0.69800440
17	4862708305/6530347008	0.74463245
18	553436255195/705277476864	0.78470712
19	1155136002965/1410554953728	0.81892308
20	2691299309615/3173748645888	0.84798754
36		0.99154188

6. *Person A rolls n dice and person B rolls m dice. What is the probability that they have a common face showing (e.g., person A rolled a 2 and person B also rolled a 2, among all their dice)?*

   We will assume 6-sided dice.

   Let X be the multiset of faces that person A rolls, and Y be the multiset of faces that person B rolls.

   We want the probability

$$P((1 \in X \text{ and } 1 \in Y) \text{ or } (2 \in X \text{ and } 2 \in Y) \text{ or} \dots).$$

Let A_i be the event " $i \in X$ and $i \in Y$ ".

Then, we want, by an application of the **inclusion-exclusion principle**,

$$P\left(\bigcup_{i=1}^{6} A_i\right) = \sum_{\substack{S\subseteq\{1,\ldots,6\}\\ S\neq\varnothing}} (-1)^{|S|-1} P\left(\bigcap_{i\in S} A_i\right).$$

Now, suppose $S \subseteq \{1,\ldots,6\}$, with $S = \{j_1,\ldots,j_{|S|}\}$. Then

$$P\left(\bigcap_{i\in S} A_i\right) = P(j_1,\ldots,j_{|S|} \in X)P(j_1,\ldots,j_{|S|} \in Y).$$

Let $\alpha_r = P(j_1,\ldots,j_r \in X)$. Then, applying inclusion-exclusion again,

$$\begin{aligned}
\alpha_r &= P(j_1,\ldots,j_r \in X)\\
&= 1 - P(j_1 \notin X \text{ or } j_2 \notin X \text{ or } \ldots)\\
&= 1 - P\left(\bigcup_{i=1}^{r} j_i \notin X\right)\\
&= 1 - \sum_{\substack{S\subseteq\{1,\ldots,r\}\\ S\neq\varnothing}} (-1)^{|S|-1} P\left(\bigcap_{i\in S} j_i \notin X\right)\\
&= 1 - \sum_{\substack{S\subseteq\{1,\ldots,r\}\\ S\neq\varnothing}} (-1)^{|S|-1} \left(1 - \frac{|S|}{6}\right)^n\\
&= 1 - \sum_{i=1}^{r} (-1)^{i-1} \left(1 - \frac{i}{6}\right)^n \binom{r}{i}.
\end{aligned}$$

Similarly, letting $\beta_r = P(j_1,\ldots,j_r \in Y)$, we have

$$\beta_r = 1 - \sum_{i=1}^{r} (-1)^{i-1} \left(1 - \frac{i}{6}\right)^m \binom{r}{i}.$$

and so

$$\begin{aligned}
P\left(\bigcap_{i\in S} A_i\right) &= \sum_{\substack{S\subseteq\{1,\ldots,6\}\\ S\neq\varnothing}} (-1)^{|S|-1} \alpha_{|S|}\beta_{|S|}\\
&= \sum_{i=1}^{6} (-1)^{i-1} \alpha_i\beta_i \binom{6}{i}.
\end{aligned}$$

Here are a few calculated values of P, the probability of a common face, for various n and m.

n	m	P	P, approx.
1	1	$1/6$	0.1666666
1	2	$11/36$	0.3055555
2	2	$37/72$	0.5138888
1	3	$91/216$	0.4212962
2	3	$851/1296$	0.6566358
3	3	$6151/7776$	0.7910236
1	4	$671/1296$	0.5177469
2	4	$1957/2592$	0.7550154
3	4	$40571/46656$	0.8695773
4	4	$86557/93312$	0.9276084
5	5	$9856951/10077696$	0.9780956
6	6	$120194317/120932352$	0.9938971

7. *On average, how many times must a* 6*-sided die be rolled until all sides appear at least once? What about for an* n*-sided die?*

   To roll until every side of the die appears, we begin by rolling once. We then roll until a different side appears. Since there are 5 different sides we could roll, this takes, on average, $\frac{1}{5/6} = \frac{6}{5}$ rolls. Then we roll until a side different from the two already rolled appears. This requires, on average, $\frac{1}{4/6} = \frac{6}{4}$ rolls. Continuing this process, and using the additive nature of expectation, we see that, on average,

   $$1 + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1} = \frac{147}{10} = 14.7$$

   rolls are needed until all 6 sides appear at least once. For an n-sided die, the number of rolls needed, on average, is

   $$1 + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1} = \sum_{i=1}^{n} \frac{n}{i} = n \sum_{i=1}^{n} \frac{1}{i}.$$

   For large n, this is approximately $n \log n$.

   (This problem is an example of what is often referred to as a Coupon Collector's problem. We can imagine that a person is trying to collect a set of n distinct coupons. Each day (say) they get a new coupon, which has a fixed probability of being one of the n types. We may then ask for the expected number of days until all n coupons have been collected. This problem is analogous to the situation in which all n coupon types are equally likely. For a more complicated version, see problem 28.)

8. *On average, how many times must a pair of* 6*-sided dice be rolled until all sides appear at least once?*

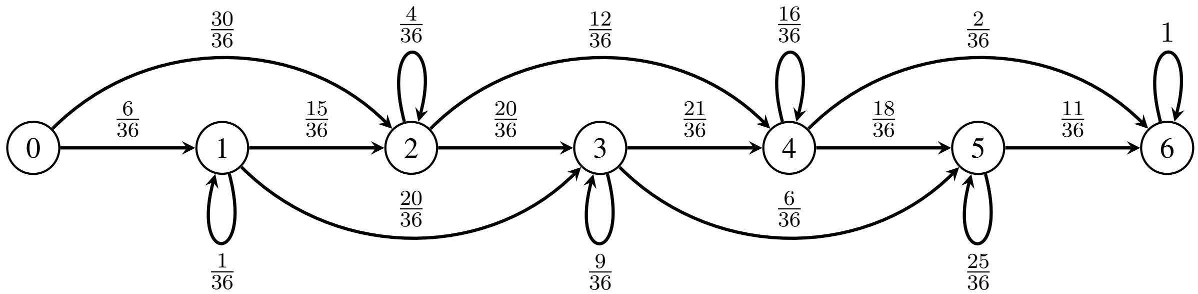
   We can solve this by treating the rolling of the dice as a *Markov process*. This means that we view our game as being always in one of a number of *states*, with a fixed probability of moving from one state to each other state in one roll of the dice.

   We can define our states by the number of sides we have seen appear so far. Thus, we starts in State 0, and we wish to end up in State 6, reaching some, or all, of States 1, 2, 3, 4 and 5 along the way.

   On the very first roll, we will move from State 0 to either State 1 or State 2. We move to State 1 with probability $\frac{6}{36}$, since this happens exactly if we roll "doubles". Otherwise, we move to State 2, so we move to State 2 from State 0 with probability $\frac{30}{36}$.

   Thus, our question can be stated thus: starting in State 0, what is the expected number of rolls until we reach State 6?

We determine the *transition probabilities*, the probability of transitioning from one state to another in one roll. We can create a diagram like this that shows the probability of moving from one state to each other state in one roll:



To solve the problem, we create a *transition matrix* for this process as follows. We let row 1 represent State 0, row 2 represent state 1, etc. Then the i,j-th entry in the matrix is the probability of transition from the row i state to the row j state in one roll (that is, from state $i-1$ to state $j-1$).

For this process, our transition matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{36} & \frac{5}{12} & \frac{5}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & \frac{5}{9} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{9} & \frac{1}{2} & \frac{1}{18} \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix Q as described in Appendix D is then

$$Q = \begin{pmatrix} 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{36} & \frac{5}{12} & \frac{5}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & \frac{5}{9} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{4}{9} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{36} \end{pmatrix}$$

The matrix $N = (I - Q)^{-1}$ as described Appendix D is then

$$\begin{pmatrix} 1 & \frac{6}{35} & \frac{57}{56} & \frac{37}{42} & \frac{43}{28} & \frac{461}{154} \\ 0 & \frac{36}{35} & \frac{27}{56} & \frac{47}{42} & \frac{41}{28} & \frac{463}{154} \\ 0 & 0 & \frac{9}{8} & \frac{5}{6} & \frac{31}{20} & \frac{329}{110} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{7}{5} & \frac{166}{55} \\ 0 & 0 & 0 & 0 & \frac{9}{5} & \frac{162}{55} \\ 0 & 0 & 0 & 0 & 0 & \frac{36}{11} \end{pmatrix}$$

Summing the first row we find the expected number of rolls until all six sides have appeared

$$= 1 + \frac{6}{35} + \frac{57}{56} + \frac{37}{42} + \frac{43}{28} + \frac{461}{154} = \frac{70219}{9240} = 7.59945887445....$$

By looking at the last entry of the first row of powers of the matrix P, we can find the probability of reaching state 6 in a given number of rolls:

rolls	probability of reaching this state in exactly this number of rolls	probability of reaching this state on or before this number of rolls
1	0	0
2	0	0
3	$5/324 \approx 0.015432099$	$5/324 \approx 0.015432099$
4	$575/5832 \approx 0.098593964$	$665/5832 \approx 0.11402606$
5	$22085/139968 \approx 0.15778607$	$38045/139968 \approx 0.27181213$
6	$313675/1889568 \approx 0.16600355$	$1654565/3779136 \approx 0.43781568$
7	$78924505/544195584 \approx 0.14502967$	$317181865/544195584 \approx 0.58284535$
8	$376014275/3265173504 \approx 0.11515905$	$2279105465/3265173504 \approx 0.69800440$
9	$61149474755/705277476864 \approx 0.086702719$	$553436255195/705277476864 \approx 0.78470712$
10	$401672322475/6347497291776 \approx 0.063280424$	$2691299309615/3173748645888 \approx 0.84798754$
11	0.045328994	0.89331653
12	0.032098630	0.92541516
13	0.022567579	0.94798274
14	0.015795289	0.96377803
15	0.011023854	0.97480189
16	0.0076798753	0.98248176
17	0.0053441053	0.98782587
18	0.0037160115	0.99154188
19	0.0025827093	0.99412459
20	0.0017945018	0.99591909
21	0.0012466057	0.99716570
22	0.00086588683	0.99803158
23	0.00060139404	0.99863298
24	0.00041767196	0.99905065

So we see that there is a less than one in a thousand chance that more than 24 rolls would be needed, for instance.

Additional question: What if we roll 3 (or greater number) dice at a time? We can answer that with another six-state Markov process; only the transition probabilities would change.

(Special thanks to Steve Hanes and Gabe for sending me this nice problem.)

9. *Suppose we roll n dice. What is the expected number of distinct faces that appear?*

   Let $\mathscr{E}$ be the sought expectation.

   I will give three distinct solutions.

   Let X be the number of distinct faces appearing in n rolls of a die. Using the **inclusion-exclusion principle**, we have the following probabilities:

$$P(X=1) = \binom{6}{1}\left(\frac{1}{6}\right)^n$$
$$P(X=2) = \binom{6}{2}\left(\left(\frac{2}{6}\right)^n - \binom{2}{1}\left(\frac{1}{6}\right)^n\right)$$
$$P(X=3) = \binom{6}{3}\left(\left(\frac{3}{6}\right)^n - \binom{3}{2}\left(\frac{2}{6}\right)^n + \binom{3}{1}\left(\frac{1}{6}\right)^n\right)$$
$$P(X=4) = \binom{6}{4}\left(\left(\frac{4}{6}\right)^n - \binom{4}{3}\left(\frac{3}{6}\right)^n + \binom{4}{2}\left(\frac{2}{6}\right)^n - \binom{4}{1}\left(\frac{1}{6}\right)^n\right)$$
$$P(X=5) = \binom{6}{5}\left(\left(\frac{5}{6}\right)^n - \binom{5}{4}\left(\frac{4}{6}\right)^n + \binom{5}{3}\left(\frac{3}{6}\right)^n - \binom{5}{2}\left(\frac{2}{6}\right)^n + \binom{5}{1}\left(\frac{1}{6}\right)^n\right)$$
$$P(X=6) = \binom{6}{6}\left(\left(\frac{6}{6}\right)^n - \binom{6}{5}\left(\frac{5}{6}\right)^n + \binom{6}{4}\left(\frac{4}{6}\right)^n - \binom{6}{3}\left(\frac{3}{6}\right)^n + \binom{6}{2}\left(\frac{2}{6}\right)^n - \binom{6}{1}\left(\frac{1}{6}\right)^n\right)$$

   These expressions determine the distribution of the number of distinct faces in n rolls.

   To find the expectation, we want

$$\mathscr{E} = \sum_{i=1}^{6} iP(X=i)$$

   and, after some chewing, this simplifies to

$$\mathscr{E} = 6 - 6\left(\frac{5}{6}\right)^n.$$

   Here's a different approach.

   The probability that the j-th roll will yield a face distinct from all previous faces rolled is

$$\frac{6\cdot 5^{j-1}}{6^j} = \left(\frac{5}{6}\right)^{j-1}$$

   since, thinking in reverse, there are 6 faces the j-th roll could be, and then 5^{j-1} ways to roll $j-1$ rolls not including that face, out of a total 6^j ways to roll j dice.

   As a result, the expected contribution from the j-th roll to the total number of distinct faces is just the probability that the j-th roll is distinct: the roll contributes 1 with that probability, and 0 otherwise.

   Using the additivity of expectation, we thus have

$$\mathscr{E} = \sum_{j=1}^{n}\left(\frac{5}{6}\right)^{j-1} = \frac{6}{5}\left(\sum_{j=0}^{n}\left(\frac{5}{6}\right)^j - 1\right) = \frac{6}{5}\left(\frac{1-\left(\frac{5}{6}\right)^{n+1}}{1-\frac{5}{6}} - 1\right) = 6 - 6\left(\frac{5}{6}\right)^n.$$

For a third solution, let X_i be a random variable defined by

$$X_i = \begin{cases} 1 & \text{if the face } i \text{ appears in } n \text{ rolls of a die,} \\ 0 & \text{otherwise.} \end{cases}$$

Let X be the number of distinct faces appearing in n rolls of a die. Then

$$X = X_1 + X_2 + X_3 + \cdots + X_6$$

and so the expected value of X is

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_6) = 6E(X_1)$$

by symmetry. Now, the probability that a 1 has appeared in n rolls is

$$P(X_1 = 1) = 1 - \left(\frac{5}{6}\right)^n$$

and so

$$E(X_1) = 1 \cdot P(X_1 = 1) = 1 - \left(\frac{5}{6}\right)^n$$

and thus the expected number of distinct faces appearing in n rolls of a die is

$$\mathscr{E} = 6E(X_1) = 6\left(1 - \left(\frac{5}{6}\right)^n\right).$$

Here's a short table of values of $\mathscr{E}$.

n	$\mathscr{E}$
1	1
2	$1.8\overline{3}$
3	$2.52\overline{7}$
4	$3.106\overline{481}$
5	3.588734...
6	3.990612...
7	4.325510...
8	4.604591...
9	4.837159...
10	5.030966...
14	5.532680...
23	5.909430...
27	5.956322...
36	5.991535...
48	5.999050...

10. *Suppose we roll n dice and keep the highest one. What is the distribution of values?*

Let's find the probability that the highest number rolled is k. Among the n dice rolled, they must all show k or less. The probability of this occurring is

$$\frac{k^n}{6^n}.$$

However, if $k > 1$, some of these rolls do not actually have any k's. That is, they are made up of only the numbers 1 through $k-1$. The probability of this occurring, for any $k \in \{1, \dots, n\}$, is

$$\frac{(k-1)^n}{6^n}$$

so the probability that the highest number rolled is k is

$$\frac{k^n - (k-1)^n}{6^n}.$$

So, for instance, the probability that, if 7 dice are rolled, the highest number to turn up will be 3 is

$$\frac{3^7 - 2^7}{6^7} = \frac{2059}{6^7} \approx 0.007355.$$

11. *Suppose we can roll a* 6*-sided die up to* n *times. At any point we can stop, and that roll becomes our "score". Our goal is to get the highest possible score, on average. How should we decide when to stop?*

If $n = 1$, there is no decision to make; on average our score is 7/2.

If $n = 2$, we want to stick if the first roll is greater than 7/2; that is, if it is 4 or greater. Otherwise, we roll again. Thus, with $n = 2$, our average score is

$$\left(\frac{1}{6}\right)4 + \left(\frac{1}{6}\right)5 + \left(\frac{1}{6}\right)6 + \left(\frac{3}{6}\right)\frac{7}{2} = \frac{17}{4} = 4.25.$$

If $n = 3$, we want to stick on the first roll if it is greater than 4.25; that is, if it is 5 or 6. Otherwise, we are in the $n = 2$ case. Thus, with $n = 3$, our average score is

$$\left(\frac{1}{6}\right)5 + \left(\frac{1}{6}\right)6 + \left(\frac{4}{6}\right)\frac{17}{4} = 4.666\ldots.$$

In general, if we let $f(n)$ be the expected value of our score with n rolls left, using s-sided dice, we have the recursion

$$f(n) = \frac{\lfloor f(n-1) \rfloor}{s} f(n-1) + \sum_{j=\lfloor f(n-1)\rfloor + 1}^{s} \frac{j}{s}$$

with $f(1) = (s+1)/2$.

We may then calculate, for $s = 6$, the following table:

n	$f(n)$
1	7/2 = 3.5
2	17/4 = 4.25
3	14/3 = 4.666..
4	89/18 = 4.944...
5	277/54 = 5.1296...
10	5.6502...
21	5.95292...
30	5.9908762...
43	5.9991472...

Thus, for a 6-sided die, we can summarize the strategy as follows:

- If there are at least 5 rolls left, stick only on 6.
- If there are 4, 3, or 2 rolls left, stick on 5 or 6.
- If there is only 1 roll left, stick on 4, 5 or 6.

12. *How many dice must be rolled to have at least a 95% chance of rolling a six? 99%? 99.9%?*

Suppose we roll n dice. The probability that none of them turn up six is

$$\left(\frac{5}{6}\right)^n$$

and so the probability that at least one is a six is

$$1-\left(\frac{5}{6}\right)^n.$$

To have a 95% chance of rolling a six, we need

$$1-\left(\frac{5}{6}\right)^n \geq 0.95$$

which yields

$$n \geq \frac{\log 0.05}{\log(5/6)} = 16.43\ldots > 16.$$

Hence, $n \geq 17$ will give at least a 95% chance of rolling at least one six. Since $\log(0.01)/\log(5/6) = 25.2585\ldots$, 26 dice are needed to have a 99% chance of rolling at least one six. Similarly, since $\log(0.001)/\log(5/6) = 37.8877\ldots$, 38 dice are needed for a 99.9% chance.

13. *How many dice must be rolled to have at least a 95% chance of rolling a one and a two? What about a one, a two, and a three? What about a one, a two, a three, a four, a five and a six?*

Solving this problem requires the use of the inclusion-exclusion principle. Of the 6^n possible rolls of n dice, 5^n have no one's, and 5^n have no two's. The number that have neither one's nor two's is *not* $5^n + 5^n$ since this would count some rolls more than once: of those 5^n rolls with no one's, some have no two's either. The number that have neither one's nor two's is 4^n, so the number of rolls that don't have at least one one, and at least one two is

$$5^n + 5^n - 4^n = 2 \cdot 5^n - 4^n$$

and so the probability of rolling a one and a two with n dice is

$$1 - \frac{2 \cdot 5^n - 4^n}{6^n}.$$

This is an increasing function of n, and by direct calculation we can show that it's greater than 0.95 for $n \geq 21$. That is, if we roll at least 21 dice, there is at least a 95% chance that there will be a one and a two among the faces that turn up.

To include three's, we need to extend the method. Of the 6^n possible rolls, there are 5^n rolls that have no one's, 5^n that have no two's, and 5^n that have no three's. There are 4^n that have neither one's nor two's, 4^n that have neither one's nor three's, and 4^n that have neither two's nor three's. In addition, there are 3^n that have no one's, two's, or three's. So, the number of rolls that don't have a one, a two, and a three is

$$5^n + 5^n + 5^n - 4^n - 4^n - 4^n + 3^n = 3 \cdot 5^n - 3 \cdot 4^n + 3^n.$$

Hence, the probability of rolling at least one one, one two, and one three is

$$1 - \frac{3 \cdot 5^n - 3 \cdot 4^n + 3^n}{6^n}.$$

This is again an increasing function of n, and it is greater than 0.95 when $n \geq 23$.

Finally, to determine the probability of rolling at least one one, two, three, four, five and six, we extend the method even further. The result is that the probability $p(n)$ of rolling at least one of every possible face is

$$p(n) = 1 - \sum_{j=1}^{5} (-1)^{(j+1)} \binom{6}{j} \left(\frac{6-j}{6}\right)^n = 1 - 6\left(\frac{1}{6}\right)^n + 15\left(\frac{1}{3}\right)^n - 20\left(\frac{1}{2}\right)^n + 15\left(\frac{2}{3}\right)^n - 6\left(\frac{5}{6}\right)^n.$$

This exceeds 0.95 when $n \geq 27$. Below is a table showing some of the probabilities for various n.

n	$p(n)$
6	0.0154...
7	0.0540...
8	0.1140...
9	0.1890...
10	0.2718...
11	0.3562...
12	0.4378...
13	0.5138...
14	0.5828...
15	0.6442...
16	0.6980...
17	0.7446...
18	0.7847...
19	0.8189...
20	0.8479...
21	0.8725...
22	0.8933...
23	0.9107...
24	0.9254...
25	0.9376...
26	0.9479...
27	0.9565...
30	0.9748...
35	0.9898...
40	0.9959...

14. *How many dice should be rolled to maximize the probability of rolling exactly one six? two sixes? n sixes?*

Suppose we roll n dice. The probability that exactly one is a six is

$$\frac{\binom{n}{1}5^{n-1}}{6^n} = \frac{n5^{n-1}}{6^n}.$$

The question is: for what value of n is this maximal? If $n > 6$ then $\frac{(n+1)5^n}{6^{n+1}} < \frac{n5^{n-1}}{6^n}$, so the maximum must occur for some $n \leq 6$. Here's a table that gives the probabilities:

n	$\frac{n5^{n-1}}{6^n}$
1	1/6 = 0.1666...
2	5/18 = 0.2777...
3	25/72 = 0.3472...
4	125/324 = 0.3858...
5	3125/7776 = 0.4018...
6	3125/7776 = 0.4018...

This shows that the maximum probability is $\frac{3125}{7776}$, and it occurs for both $n = 5$ and $n = 6$.

For two sixes, the calculation is similar. The probability of exactly two sixes when rolling n dice is

$$\frac{\binom{n}{2}5^{n-2}}{6^n} = \frac{n(n-1)5^{n-2}}{2 \cdot 6^n}$$

A quick calculation shows that this is maximal for $n = 12$ or $n = 11$.

It seems that for n sixes, the maximal probability occurs with $6n$ and $6n - 1$ dice. I'll let you prove that.

15. *Suppose we roll a fair die 100 times. What is the probability of a run of at least 10 sixes?*

We will consider this problem generally.

Let p_n be the probability of a run of at least r successes in n throws. Let α be the probability of success on any one throw (so when throwing a single fair die, $\alpha = 1/6$.)

Clearly $p_n = 0$ if $n < r$.

We can determine p_{n+1} in terms of p_n and p_{n-r}. There are two ways that a run of r can happen in $n + 1$ throws. Either (a) there is a run of r in the first n throws, or (b) there is not, and the final r throws of the $n + 1$ are all successes.

The probability of (a) occurring is p_n.

To calculate the probability of (b), first note that for (b) to occur, three things have to happen:

(a) There is no run of length r in the first $n - r$ throws; this happens with probability $1 - p_{n-r}$.

(b) On throw number $n - r + 1$, we do not get a success. If we did, then we would have a run of r successes in the first n throws (since the final r throws are all successes). The probability here is $1 - \alpha$.

(c) The final r throws are all successes. The probability of this is α^r.

Since these three events are independent, we find that

$$p_{n+1} = p_n + (1 - p_{n-r})(1 - \alpha)\alpha^r.$$

Since r and α are fixed, this is a linear recurrence equation, and we have initial conditions

$$p_0 = p_1 = ... = p_{r-1} = 0, \text{ and } p_r = \alpha^r.$$

If we take $n = r$, we find

$$p_{r+1} = p_r + (1 - p_0)(1 - \alpha)\alpha^r = \alpha^r + (1 - \alpha)\alpha^r = \alpha^r(2 - \alpha).$$

and then

$$p_{r+2} = p_{r+1} + (1 - p_1)(1 - \alpha)\alpha^r = \alpha^r(3 - 2\alpha).$$

Similarly, if $r > 2$ then

$$p_{r+3} = p_{r+2} + (1 - p_2)(1 - \alpha)\alpha^r = \alpha^r(3 - 2\alpha) + (1 - \alpha)\alpha^r = \alpha^r(4 - 3\alpha).$$

So, for instance, the probability of a run of at least 3 sixes when a die is thrown 5 times is (with $r = 3$ and $\alpha = 1/6$)

$$p_5 = \left(\frac{1}{6}\right)^3 \left(3 - \frac{2}{6}\right) = \frac{1}{81}$$

and if the die is thrown 6 times the probability is

$$p_6 = \left(\frac{1}{6}\right)^3 \left(4 - \frac{3}{6}\right) = \frac{7}{432} = \frac{1}{61.714...}.$$

With this recurrence equation, we can calculate an expression for p_{r+3}, p_{r+4}, etc.

To answer the question "what is the probability of a run of 10 sixes in 100 throws of a fair die?" we wish to calculate p_{100} with $\alpha = 1/6$ and $r = 10$. Using a free computer algebra system (like PARI/GP), we can determine that, with $r = 10$ and $\alpha = 1/6$,

$$\begin{aligned} p_{100} &= -10\alpha^{99} + 135\alpha^{98} - 720\alpha^{97} + 2100\alpha^{96} - 3780\alpha^{95} + 4410\alpha^{94} - 3360\alpha^{93} + 1620\alpha^{92} - 450\alpha^{91} \\ &\quad + 55\alpha^{90} - 125970\alpha^{88} + 1085280\alpha^{87} - 4069800\alpha^{86} + 8682240\alpha^{85} - 11531100\alpha^{84} + 9767520\alpha^{83} \\ &\quad - 5155080\alpha^{82} + 1550400\alpha^{81} - 203490\alpha^{80} - 2035800\alpha^{77} + 14844375\alpha^{76} - 46314450\alpha^{75} \\ &\quad + 80159625\alpha^{74} - 83128500\alpha^{73} + 51658425\alpha^{72} - 17813250\alpha^{71} + 2629575\alpha^{70} - 3838380\alpha^{66} \\ &\quad + 23688288\alpha^{65} - 60865740\alpha^{64} + 83347680\alpha^{63} - 64155780\alpha^{62} + 26320320\alpha^{61} - 4496388\alpha^{60} \\ &\quad - 2118760\alpha^{55} + 10824100\alpha^{54} - 22108800\alpha^{53} + 22569400\alpha^{52} - 11515000\alpha^{51} + 2349060\alpha^{50} \\ &\quad - 487635\alpha^{44} + 1984760\alpha^{43} - 3028470\alpha^{42} + 2053200\alpha^{41} - 521855\alpha^{40} - 54740\alpha^{33} + 166635\alpha^{32} \\ &\quad - 169050\alpha^{31} + 57155\alpha^{30} - 3160\alpha^{22} + 6400\alpha^{21} - 3240\alpha^{20} - 90\alpha^{11} + 91\alpha^{10} \\ &= \frac{2138428277869029245997109282919411017852189744280011307296262359092389}{1701350582031434651293464237390775574315478412689986644643416579087232139264} \\ &= 0.00000125690042984... = \frac{1}{795607.97...}. \end{aligned}$$

16. *Suppose we roll a fair die until some face has appeared twice. For instance, we might have a run of rolls 12545 or 636. How many rolls on average would we make? What if we roll until a face has appeared three times?*

For the first part of the question, we can enumerate easily the possibilities. Let X be the number of rolls made until a face has appeared twice. We would like to know $P(X = x)$ for $2 \leq x \leq 7$.

In the $X = 2$ case, our run of rolls must have the form AA, where $1 \leq A \leq 6$. So there are 6 such runs, out of 6^2 possible. Hence,

$$P(X = 2) = \frac{6}{6^2} = \frac{1}{6}.$$

In the $X = 3$ case, our run of rolls must have the form ABA or BAA, and so

$$P(X = 3) = 2\frac{6 \cdot 5}{6^3} = \frac{5}{18}.$$

In the $X = 4$ case, our run of rolls must have the form $ABCA$, $BACA$, or $BCAA$, and so

$$P(X = 4) = 3\frac{6 \cdot 5 \cdot 4}{6^4} = \frac{5}{18}.$$

Similarly, we have

$$P(X = 5) = 4\frac{6 \cdot 5 \cdot 4 \cdot 3}{6^5} = \frac{5}{27}$$
$$P(X = 6) = 5\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^6} = \frac{25}{324}$$
$$P(X = 7) = 6\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6^7} = \frac{5}{324}$$

Thus we see that $X = 3$ and $X = 4$ are tied as the most likely, and the expected number of rolls is

$$\sum_{i=2}^{7} iP(X = i) = \frac{1223}{324} = 3.7746913580246....$$

When rolling until a face appears three times, things are a little more complex. For fun, I thought of treating this as a Markov chain. The number of states is quite large: as we roll, we keep track of the number of 1's, 2's, etc. that have been rolled. Hence there will be $3^6 = 729$ states to consider, plus the absorbing state, for a total of 730 states. We can map the number of appearances of each face to a state by a function as follows. Suppose the number of appearances of face i is a_i. Then we can number the current state as

$$S = 1 + a_1 + 3a_2 + 3^2a_3 + 3^3a_4 + 3^4a_5 + 3^5a_6$$

Then, we create a transition matrix to express the probability of going from state S to state T, for all possible states. Here is some GP/PARI code which does this:

```
\\ define a function to map the vector of face counts to a state number
state(a,b,c,d,e,f)=1+a+3*b+9*c+27*d+81*e+243*f;\
\\ initialize a matrix for the transition probabilities
A=matrix(730,730);\
\\ generate the probabilities and put them in the matrix
for(a1=0,2,for(a2=0,2,for(a3=0,2,for(a4=0,2,for(a5=0,2,for(a6=0,2,\
print(a1);\
\\ v is the vector of counts
v=vector(6);v[1]=a1;v[2]=a2;v[3]=a3;v[4]=a4;v[5]=a5;v[6]=a6;\
\\ s is the state
s=state(a1,a2,a3,a4,a5,a6);\
\\ look at how many face counts are equal to 2,
\\ since there is a 1/6 chance for each
\\ such face that we'll go to the absorbing state from here
c=0;for(i=1,6,if(v[i]==2,c=c+1));\
print(c);\
\\ create a new vector w of the counts,
\\ then increase each face (with count<2) by one,
\\ and see where we go, give 1/6 probability of going to that state
w=vector(6);for(i=1,6,w[i]=v[i]);\
```

```
A[s+1,0+1]=c/6;\
for(i=1,6,for(j=1,6,w[j]=v[j]);\
if(w[i]<2,w[i]=w[i]+1;ss=state(w[1],w[2],w[3],w[4],w[5],w[6]);\
A[s+1,ss+1]=1/6;\
))\
))))))
```

Once we have the transition matrix A, we can calculate A^n for $n = 1, ..., 13$ and determine the probabilities of ending in exactly n rolls:

n	$P(X = n)$	$P(X \leq n)$
1	0	0
2	0	0
3	$\frac{1}{36} = 0.02\bar{7}$	$\frac{1}{36} = 0.02\bar{7}$
4	$\frac{5}{72} = 0.069\bar{4}$	$\frac{7}{72} = 0.97\bar{2}$
5	$\frac{25}{216} = 0.11574\bar{0}7$	$\frac{23}{108} = 0.212962...$
6	$\frac{25}{162} = 0.154320...$	$\frac{119}{324} = 0.367283...$
7	$\frac{25}{144} = 0.1736\bar{1}$	$\frac{701}{1296} = 0.540895...$
8	$\frac{1295}{7776} = 0.166538...$	$\frac{5501}{7776} = 0.707433...$
9	$\frac{175}{1296} = 0.135030...$	$\frac{6551}{7776} = 0.842463...$
10	$\frac{175}{1944} = 0.0900205...$	$\frac{2417}{2592} = 0.932484...$
11	$\frac{4375}{93312} = 0.0468857...$	$\frac{91387}{93312} = 0.979370...$
12	$\frac{9625}{559872} = 0.0171914...$	$\frac{557947}{559872} = 0.996561...$
13	$\frac{1925}{559872} = 0.00343828...$	1

We find the expected number of rolls to be

$$\sum_{i=1}^{13} iP(X = i) = \frac{4084571}{559872} = 7.2955443387059899....$$

Additional questions: what if we roll until a face appears 4 times, or 5 times, etc?

17. *Suppose we roll a fair die 10 times. What is the probability that the sequence of rolls is non-decreasing (i.e., the next roll is never less than the current roll)?*

For example, the sequence $\{1, 2, 2, 2, 3, 4, 5, 5, 5, 6\}$ is a non-decreasing sequence.

The total number of possible roll sequences is 6^{10}. How many of these are non-decreasing?

An excellent observation is that every non-decreasing sequence is equivalent to a "histogram" or vector which gives the number of times each face appears.

For example, the sequence $\{1, 2, 2, 2, 3, 4, 5, 5, 5, 6\}$ is equivalent to the vector $\langle 1, 3, 1, 1, 3, 1\rangle$. By equivalent, I mean that there is a one-to-one correspondence between the sequences and vectors. So, counting one is equivalent to counting the other.

Thus, we wish to count how many ways can 10 indistinguishable things be placed into 6 bins, where we allow for zero items to be placed in some bins.

To count *that*, we observe that this is equivalent to the number of ways to place 16 indistinguishable things into 6 bins, where each bin must contain at least one item. Subtracting one from each bin will give us a vector of the previous sort.

To count this, we can use the stars-and-bars method. Putting 16 things into 6 bins is equivalent to putting 5 bars among 16 stars, such that there is at most one bar between any two stars. For instance, this choice of bars:

$$* * *| * * * * * | * *| * | * * * *|*$$

represents the vector $\langle 3, 5, 2, 1, 4, 1\rangle$ which, if we subtract one from each component yields the vector $\langle 2, 4, 1, 0, 3, 0\rangle$ which corresponds to the rolled sequence $1, 1, 2, 2, 2, 2, 3, 5, 5, 5$.

Since there are 16 stars, there are 15 places for bars, and hence the number of such sequences is

$$\binom{15}{5} = 3003$$

Thus, the probability of rolling such a sequence is a very low

$$\frac{3003}{6^{10}} = \frac{1001}{20155392} = 0.0000496641295788... = \frac{1}{20135.25674...}$$

Generally, for a sequence of n rolls, the probability is

$$p_n = \frac{\binom{n+6-1}{5}}{6^n}$$

Here is a table of some values

n	p_n
1	1
2	$\frac{7}{12} = 0.58\bar{3}$
3	$\frac{7}{27} = 0.\overline{259}$
4	$\frac{7}{72} = 0.097\bar{2}$
5	$\frac{7}{216} = 0.032\overline{407}$
6	$\frac{77}{7776} = 0.00990226...$
7	$\frac{11}{3888} = 0.00282921...$
8	$\frac{143}{186624} = 0.000766246...$
9	$\frac{1001}{5038848} = 0.000198656...$
10	$\frac{1001}{20155293} = 0.0000496641...$

The value of p_{12} is greater than one-in-a-million, but p_{13} is less.

18. *Suppose a pair of dice are thrown, and then thrown again. What is the probability that the faces appearing on the second throw are the same as the first?*

    What if three dice are used? Or six?

    We may consider two cases. If two dice are thrown, the result will either be two different faces, or the same face twice. We may notate these two cases as “AB” and “AA” (this will be useful later). The probability that two different faces will appear is

$$\frac{6 \cdot 5}{6^2} = \frac{5}{6}$$

and the probability that the second throw will be the same as the first in this case is

$$\frac{2}{6^2}.$$

Thus, the probability that the second roll will repeat the first in this way is

$$\frac{6 \cdot 5 \cdot 2}{6^4} = \frac{5}{108}.$$

The other possibility of of rolling doubles. This case gives a probability of

$$\left(\frac{6}{6^2}\right)\left(\frac{1}{6^2}\right) = \frac{6}{6^4} = \frac{1}{216}$$

of occurring. Adding together, we find the probability of the second throw being identical to the first is

$$\frac{5}{108} + \frac{1}{216} = \frac{11}{216} = 0.0509259....$$

If we throw three dice, there are more cases to consider. These cases may be expressed as AAA, AAB, and ABC. (For example, throwing $\{1, 3, 3\}$ would be an example of the AAB case, while $\{2, 4, 5\}$ would be an example of the ABC case.) The probability of repeating via each case is as follows:

AAA	$\left(\frac{\binom{6}{1}}{6^3}\right)\left(\frac{1}{6^3}\right) = \frac{6}{6^6}$
AAB	$\left(\frac{\binom{6}{1}\binom{5}{1}\binom{3}{1}}{6^3}\right)\left(\frac{3}{6^3}\right) = \frac{270}{6^6}$
ABC	$\left(\frac{\binom{6}{3}\cdot 3!}{6^3}\right)\left(\frac{3!}{6^3}\right) = \frac{720}{6^6}$

The first factor in each case is the probability of rolling that case, and the second is the probability of rolling the same set of faces a second time.

Adding these, we see that the probability of repeating with three dice is

$$\frac{996}{6^6} = \frac{83}{3888} = 0.02134773662551....$$

For six dice, the problem is similar, just with more cases. Here is the calculation:

AAAAAA	$\left(\frac{6}{6^6}\right)\left(\frac{1}{6^6}\right)$	$= \frac{6}{6^{12}}$
AAAAAB	$\left(\frac{6\cdot5\cdot\binom{6}{1}}{6^6}\right)\left(\frac{6}{6^6}\right)$	$= \frac{1080}{6^{12}}$
AAAABB	$\left(\frac{6\cdot5\cdot\binom{6}{2}}{6^6}\right)\left(\frac{\binom{6}{2}}{6^6}\right)$	$= \frac{6750}{6^{12}}$
AAAABC	$\left(\frac{6\cdot5\cdot4\cdot(6\cdot5)}{6^6}\right)\left(\frac{\binom{6}{2}}{6^6}\right)$	$= \frac{54000}{6^{12}}$
AAABBB	$\left(\frac{\binom{6}{2}\binom{6}{3}}{6^6}\right)\left(\frac{\binom{6}{3}}{6^6}\right)$	$= \frac{6000}{6^{12}}$
AAABBC	$\left(\frac{6\cdot5\cdot4\cdot\binom{6}{2}\cdot\binom{4}{1}}{6^6}\right)\left(\binom{6}{2}\binom{4}{1}\right)$	$= \frac{432000}{6^{12}}$
AAABCD	$\left(\frac{6\cdot5\cdot4\cdot3\cdot\binom{6}{3}}{6^6}\right)\left(\frac{6\cdot5\cdot4}{6^6}\right)$	$= \frac{864000}{6^{12}}$
AABBCC	$\left(\frac{6\cdot5\cdot4\cdot\frac{\binom{6}{2}\binom{4}{2}}{3!}}{6^6}\right)\left(\frac{\binom{6}{2}\binom{4}{2}}{6^6}\right)$	$= \frac{162000}{6^{12}}$
AABBCD	$\left(\frac{6\cdot5\cdot4\cdot3\cdot\frac{\binom{6}{2}\binom{4}{2}\binom{2}{1}}{2\cdot2}}{6^6}\right)\left(\frac{\binom{6}{2}\binom{4}{2}\binom{2}{1}}{6^6}\right)$	$= \frac{2916000}{6^{12}}$
AABCDE	$\left(\frac{6\cdot5\cdot4\cdot3\cdot2\frac{\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1}}{4!}}{6^6}\right)\left(\frac{\binom{6}{2}\binom{4}{1}\binom{3}{1}\binom{2}{1}}{6^6}\right)$	$= \frac{3888000}{6^{12}}$
ABCDEF	$\left(\frac{6!}{6^6}\right)\left(\frac{6!}{6^6}\right)$	$= \frac{518400}{6^{12}}$

(For example, rolling $\{1, 2, 3, 3, 5, 5\}$ would be an example of the AABBCD case.)

The first factor in each case is the probability of rolling that case, and the second is the probability of rolling the same set of faces a second time.

Adding the probabilities for all cases gives a total probability of

$$\frac{8848236}{6^{12}} = \frac{737353}{181398528} = 0.004064823502...$$

19. *A single die is rolled until a run of six different faces appears. For example, one might roll the sequence 535463261536435344151612534 with only the last six rolls all distinct. What is the expected number of rolls?*

We may solve this by creating a set of linear recurrence equations. Let E_i be the expected number of rolls from a point where the last i rolls were distinct. We seek E_0. We have then

$$E_0 = 1 + E_1 \tag{3.6}$$

$$E_1 = 1 + \frac{1}{6}E_1 + \frac{5}{6}E_2 \tag{3.7}$$

$$E_2 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{4}{6}E_3 \tag{3.8}$$

$$E_3 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{3}{6}E_4 \tag{3.9}$$

$$E_4 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{1}{6}E_4 + \frac{2}{6}E_5 \tag{3.10}$$

$$E_5 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{1}{6}E_4 + \frac{1}{6}E_5 + \frac{1}{6}E_6 \tag{3.11}$$

$$E_6 = 0 \tag{3.12}$$

The last zero rolls are distinct only before the rolls have started, so $E_0 = 1 + E_1$ since there must be a roll, and that takes us to the state where the last 1 roll is distinct. Then another roll occurs; at this point, with probability 1/6 the roll is the same as the last roll, and so we remain in the same state, or, with probability 5/6, a different face appears, and then the last two rolls are distinct. The pattern continues this way.

Thus we have a system of seven linear equations in seven unknowns, which is solvable via many methods. The result is

$$\begin{aligned}
E_0 &= \frac{416}{5} = 83.2 \\
E_1 &= \frac{411}{5} = 82.2 \\
E_2 &= 81 \\
E_3 &= \frac{396}{5} = 79.2 \\
E_4 &= \frac{378}{5} = 75.6 \\
E_5 &= \frac{324}{5} = 64.8
\end{aligned}$$

Thus, on average, it will take 83.2 rolls before getting a run of six distinct faces.

20. *What is the most probable: rolling at least one six with six dice, at least two sixes with twelve dice, or at least three sixes with eighteen dice? (This is an old problem, frequently connected with Isaac Newton.)*

One way to solve this is to simply calculate the probability of each. The probability of rolling exactly m sixes when rolling r six-sided dice is

$$\binom{r}{m}\frac{5^{r-m}}{6^r}$$

so the probability of rolling at least m sixes when rolling r six-sided dice is

$$p(m, r) = \sum_{i=m}^{r} \binom{r}{i}\frac{5^{r-i}}{6^r}.$$

Grinding through the calculations yields

$$p(1, 6) = \frac{31031}{46656} \approx 0.66510202331961591221$$

$$p(2, 12) = \frac{1346704211}{2176782336} \approx 0.61866737373230871348$$

$$p(3, 18) = \frac{15166600495229}{25389989167104} \approx 0.59734568594772319497$$

so that we see that the six dice case is the clear winner.

21. *Suppose we roll n dice, remove all the dice that come up 1, and roll the rest again. If we repeat this process, eventually all the dice will be eliminated. How many rolls, on average, will we make? Show, for instance, that on average fewer than $O(\log n)$ throws occur.*

We expect that, on average, 5/6 of the dice will be left after each throw. So, after k throws, we expect to have $n\left(\frac{5}{6}\right)^k$ dice left. When this is less than 2, we have, on average less than 6 throws left, so the number of throws should be, on average, something less than a constant time $\log n$.

Let M_n be the expected number of throws until all dice are eliminated. Then, thinking in terms of a Markov chain, we have the recurrence formula

$$M_n = \frac{1}{6^n} + \left(\frac{5}{6}\right)^n (1+M_n) + \sum_{j=1}^{n-1} (1+M_j) \binom{n}{n-j} \frac{5^j}{6^n}$$

which allows us to solve for M_n:

$$M_n = \frac{1 + 5^n + \sum_{j=1}^{n-1} (1+M_j) \binom{n}{n-j} 5^j}{6^n - 5^n}$$

Here are a few values of M_n.

n	M_n
1	6
2	8.72727272727273
3	10.5554445554446
4	11.9266962545651
5	13.0236615075553
6	13.9377966973204
7	14.7213415962620
8	15.4069434778816
9	16.0163673664838
10	16.5648488612594
15	18.6998719821123
20	20.2329362496041
30	22.4117651317294
40	23.9670168145374
50	25.1773086926527

We see that M_n increases quite slowly, another suggestion that $M_n = O(\log n)$. To show this, suppose $M_j < C \log j$ for all $2 \leq j < n$. Then we have

$$M_n < \frac{1 + 5^n + \max\{1+6, 1 + C\log(n-1)\} \sum_{j=1}^{n-1} \binom{n}{n-j} 5^j}{6^n - 5^n}$$

$$= \frac{1 + 5^n + C\log(n-1)(6^n - 5^n - 1)}{6^n - 5^n} = C\left(1 - \frac{1}{6^n - 5^n}\right)\log(n-1) + \frac{1+5^n}{6^n - 5^n} < C \log n$$

if and only if

$$\left(1 - \frac{1}{6^n - 5^n}\right)\frac{\log(n-1)}{\log n} + \frac{1+5^n}{C\log n(6^n - 5^n)} < 1$$

Since $M_2/\log 2 < 13$, we may suppose $C = 13$. It is not hard to show the above inequality holds for all n, and hence $M_n < 13\log n$ for all $n \geq 2$.

22. *Suppose we roll a die $6k$ times. What is the probability that each possible face comes up an equal number of times (i.e., k times)? Find an asymptotic expression for this probability in terms of k.*

    In the $6k$ rolls, we want k of them to appear as the face "1". There are

    $$\binom{6k}{k}$$

    ways this can occur. There are then

    $$\binom{5k}{k}$$

    ways for k 2s to occur among the $6k - k = 5k$ remaining spots.

    Continuing, we can conclude that there are

    $$\binom{6k}{k}\binom{5k}{k}\binom{4k}{k}\binom{3k}{k}\binom{2k}{k}\binom{k}{k}$$

    ways to rolls an equal number of each face when rolling $6k$ times.

    Hence the probability of this happening is

    $$\frac{\binom{6k}{k}\binom{5k}{k}\binom{4k}{k}\binom{3k}{k}\binom{2k}{k}\binom{k}{k}}{6^{6k}} = \frac{(6k)!}{(k!)^6 6^{6k}}$$

    after simplification.

    By applying Stirling's approximation for the factorial,

    $$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

    we can approximate the probability as

    $$\begin{aligned}\frac{(6k)!}{(k!)^6 6^{6k}} &\approx \frac{\sqrt{2\pi\cdot 6k}\left(\frac{6k}{e}\right)^{6k}}{\left(\sqrt{2\pi k}\left(\frac{k}{e}\right)^k\right)^6 6^{6k}} \\ &= \frac{2\pi\cdot 6k}{(2\pi k)^3} \\ &= \frac{\sqrt{6}}{(2\pi)^{5/2}}k^{-5/2}\end{aligned}$$

    Let $P(k)$ be the probability of rolling an equal number of all faces in $6k$ rolls of a die. Then we have

    $$\lim_{k\to\infty} \frac{P(k)}{\frac{\sqrt{6}}{(2\pi)^{5/2}}k^{-5/2}} = 1.$$

23. *Call a "consecutive difference" the absolute value of the difference between two consecutive rolls of a die. For example, the sequence of rolls 14351 has the corresponding sequence of consecutive differences $3, 1, 2, 4$. What is the expected number of times we need to roll a die until all 6 consecutive differences have appeared?*

    We can solve this with a Markov chain.

Here, we can define the state to be a pair (d, S), where d is the latest roll of the die, and S is the set of differences already achieved.

For computation, we can define a vector $\langle a_1, a_2, a_3, a_4, a_5, a_6 \rangle$ with $a_i = 1$ if $i \in S$, and $a_i = 0$ if $i \notin S$.

We can then convert the state pair (d, S) into a unique non-negative integer by

$$s = 64(d-1) + \sum_{i=1}^{6} 2^{i-1} a_i.$$

Thus, each state corresponds to a row of a 384×384 matrix.

We may define a 385th row (and column) corresponding to the absorbing, "finished" state, since once we have achieved all differences, it does not matter what additional rolls occur.

We then calculate, for each state, the new state attained by rolling $1, 2, \ldots 6$ on the next roll and add this information to create the transition matrix.

Here is some GP code that does this:

```
{
A=vector(6);
M=matrix(64*6+1,64*6+1);
for(a0=0,1,for(a1=0,1,for(a2=0,1,for(a3=0,1,for(a4=0,1,for(a5=0,1,
for(d=1,6,
A=[a0,a1,a2,a3,a4,a5];
state1=64*(d-1)+sum(i=1,6,2^(i-1)*A[i]);
for(e=1,6,diff=abs(d-e);B=A;B[diff+1]=1;if(B==[1,1,1,1,1,1],
M[state1+1,64*6+1]=M[state1+1,64*6+1]+1/6,
state2=64*(e-1)+sum(i=1,6,2^(i-1)*B[i]);
M[state1+1,state2+1]=M[state1+1,state2+1]+1/6;
) )
)
))))));
M[385,385]=1;
}
```

We then let the matrix Q be M with the last row and column removed, and calculate $N = (I - Q)^{-1}$ (as described in Appendix D).

If we sum rows 1, $1 + 64$, $1 + 64 \cdot 2$, etc., we find that the expected number of rolls needed after starting the sequence with a roll of d are (approximately)

1	23.77122103041073289277579517
2	25.27138286232360165381869693
3	25.50271996489307011817416107
4	25.50271996489307011817416107
5	25.27138286232360165381869693
6	23.77122103041073289277579517

and since each of these starting rolls is equally likely, the average of these, plus 1, yields the overall expected number of rolls needed until all absolute differences have occurred:

$$\frac{672875275767847611958914137}{26031560987606728347794000} \approx 25.84844128587580155492288439.$$

Note that in the table above, a 1 or 6 on the first roll markedly reduces the expected number of rolls, since we must roll a 1 and a 6 consecutively at some point, whereas all other differences can be achieved in more than one way.

3.2 Dice Sums

24. *Show that the probability of rolling 14 is the same whether we throw 3 dice or 5 dice.*

This seems like a tedious calculation, and it is. To save some trouble, we can use a computer algebra system to determine the coefficient of x^{14} in the polynomials $(x + x^2 + x^3 + x^4 + x^5 + x^6)^3$ and $(x + x^2 + x^3 + x^4 + x^5 + x^6)^5$ (see Appendix C for an explanation of this method). They are 15 and 540, respectively, and so the probability in question is $\dfrac{15}{6^3} = \dfrac{540}{6^5} = \dfrac{5}{72}$.

Are there other examples of this phenomenon?

Yes. Let $p_d(t, n)$ be the probability of rolling a sum of t with n $d-$sided dice. Then:

- $p_3(5, 2) = p_3(5, 3) = \dfrac{2}{9}$
- $p_3(10, 4) = p_3(10, 6) = \dfrac{10}{81}$
- $p_4(9, 3) = p_4(9, 4) = \dfrac{5}{32}$
- $p_6(14, 3) = p_6(14, 5) = \dfrac{5}{72}$
- $p_9(15, 2) = p_9(15, 4) = \dfrac{4}{81}$
- $p_{20}(27, 2) = p_{20}(27, 3) = \dfrac{7}{200}$

Questions: Are there others? Can we find *all* of them?

25. *Suppose we roll n dice and sum the highest 3. What is the probability that the sum is 18?*

In order for the sum to be 18, there must be at least three 6's among the n dice. So, we could calculate probability that there are 3,4,5,. . . ,n 6's among the n dice. The sum of these probabilities would be the probability of rolling 18. Since n could be much greater than 3, an easier way to solve this problem is to calculate the probability that the sum is *not* 18, and then subtract this probability from 1. To get a sum that is not 18, there must be 0, 1 or 2 6's among the n dice. We calculate the probability of each occurrence:

zero 6's: the probability is $\dfrac{5^n}{6^n}$

one 6: the probability is $\dfrac{n5^{n-1}}{6^n}$

two 6's: the probability is $\dfrac{\binom{n}{2}5^{n-2}}{6^n}$

Hence, the probability of rolling a sum of 18 is

$$1 - \left(\frac{5^n}{6^n} + \frac{n5^{n-1}}{6^n} + \frac{\binom{n}{2}5^{n-2}}{6^n}\right) = 1 - \left(\frac{5}{6}\right)^n \left(1 + \frac{9}{50}n + \frac{1}{50}n^2\right) = p(n)$$

say. Then, for example, $p(1) = p(2) = 0$, $p(3) = 1/216$, $p(4) = 7/432$, and $p(5) = 23/648$.

26. *Four fair, 6-sided dice are rolled. The highest three are summed. What is the distribution of the sum?*

This is a quick calculation with a tiny bit of coding. In PARI/GP, the computation looks like this:

```
gp > A=vector(20);
gp > for(i=1,6,for(j=1,6,for(k=1,6,for(m=1,6,
          s=i+j+k+m-min(min(i,j),min(k,m));A[s]=A[s]+1))))
gp > A
[0, 0, 1, 4, 10, 21, 38, 62, 91, 122, 148, 167, 172, 160, 131, 94, 54, 21, 0, 0]
```

(The funny min(min(i,j),min(k,m)) bit is there because the default min function only works with two values, and we want the minimum of i, j, k and m.)

If we define $A(n)$ to be the number of rolls out of 6^4 which yield a sum of n, we have the following table:

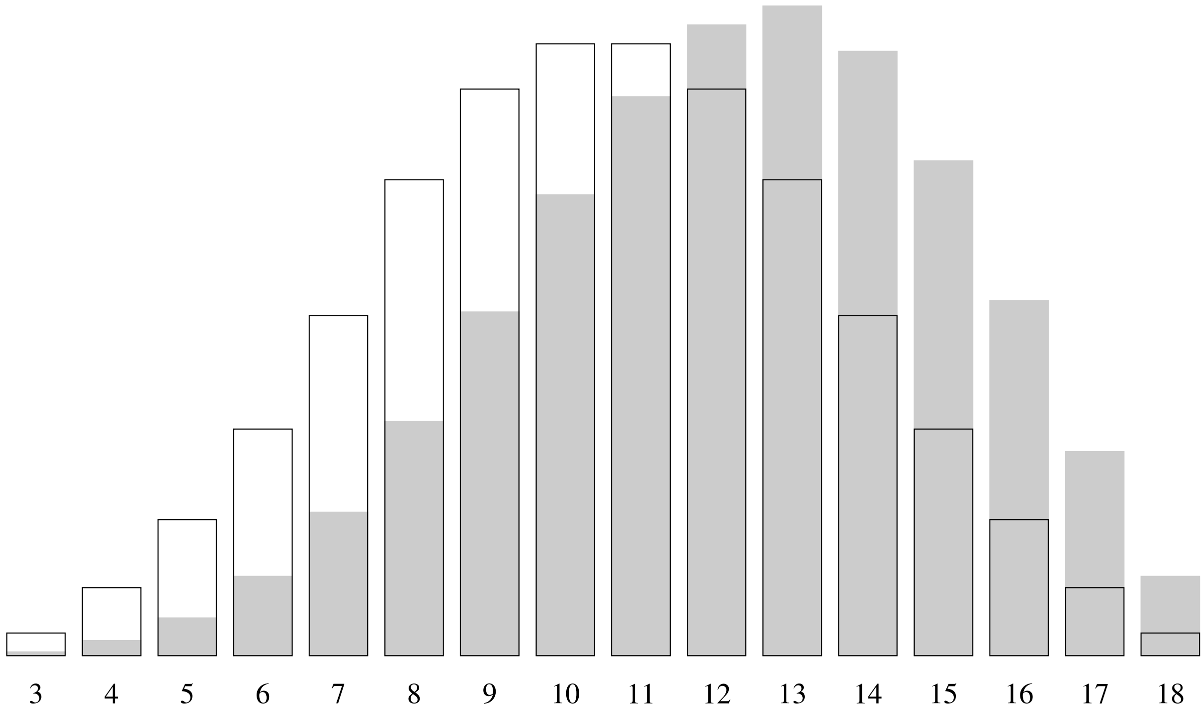
n	A(n)	probability
3	1	$1/1296 \approx 0.00077$
4	4	$1/324 \approx 0.00308$
5	10	$5/648 \approx 0.00771$
6	21	$7/432 \approx 0.01620$
7	38	$19/648 \approx 0.02932$
8	62	$31/648 \approx 0.04783$
9	91	$91/1296 \approx 0.07021$
10	122	$61/648 \approx 0.09413$
11	148	$37/324 \approx 0.11419$
12	167	$167/1296 \approx 0.12885$
13	172	$43/324 \approx 0.13271$
14	160	$10/81 \approx 0.12345$
15	131	$131/1296 \approx 0.10108$
16	94	$47/648 \approx 0.07253$
17	54	$1/24 \approx 0.04166$
18	21	$7/432 \approx 0.01620$

How does this compare to the distribution of the sums of three dice?

We see the most likely roll is 13, compared to a tie for 10 and 11 with a simple roll of three dice.

The mean roll here is $\frac{1}{6^4}\sum_{i=3}^{18} iA(i) = \frac{15869}{1296} \approx 12.2445987654...$ compared to a mean of 10.5 for a simple roll of three dice.

Here is a histogram comparing the distribution of sums for the "roll four, drop one" and the simple roll three methods.



27. *A fair, n-sided die is rolled until a roll of k or greater appears. All rolls are summed. What is the expected value of the sum?*

The probability that any roll is greater than or equal to k is

$$\frac{n+1-k}{n}$$

so the expected number of rolls until a roll of k or greater is

$$\frac{n}{n+1-k}.$$

All but the last one of these rolls is less than k, so the expected value of the sum of these rolls is

$$\left(\frac{n}{n+1-k}-1\right)\frac{1+(k-1)}{2}.$$

We add to this the expected value of the final roll

$$\frac{k+n}{2}$$

and so the expectation of the entire sum is

$$\left(\frac{n}{n+1-k}-1\right)\frac{1+(k-1)}{2}+\frac{k+n}{2}=\frac{n^2+n}{2n-2k+2}$$

We can also argue as follows. Let E be the expected value of the sum. One the first roll, the sum is either less than k or it is k or greater. If it is less, then we can express E like this:

$$E \;=\; \frac{k-1}{n}(\ \{\text{expected value of roll} < k\}\ + E) + \frac{n+1-k}{n}(\ \text{expected value of roll} \geq k)$$

$$= \quad \frac{k-1}{n}\left(\frac{k}{2}+E\right)+\frac{n+1-k}{n}\left(\frac{k+n}{2}\right)$$

From this we have

$$2n(1-\frac{k-1}{n})E = k(k-1)+(n+1-k)(k+n) = n^2+n$$

from which we find

$$E = \frac{n^2+n}{2n-2k+2}.$$

More explicitly, and without the assumption of the uniform distribution of the dice values, we may write

$$\begin{aligned} E &= \sum_{i=1}^{k-1}\left(\frac{1}{n}i+E\right)+\sum_{i=k}^{n}\frac{1}{n}i \\ &= \frac{k-1}{n}\left(\frac{k}{2}+E\right)+\frac{n+1-k}{n}\left(\frac{k+n}{2}\right) \end{aligned}$$

and the rest follows as above.

28. *A pair of dice is rolled repeatedly. What is the expected number of rolls until all eleven possible sums have appeared? What if three dice are rolled until all sixteen possible sums have appeared?*

This is an example of a so-called **coupon collector's problem**. We imagine a person is seeking to complete a set of n distinct "coupons". Each day they get one coupon (from a presumably infinite source), and we wish to know how many days are expected in order to get the complete set.

If the probability of each coupon appearing is the same as all others, the problem is fairly simple (see problem 7). When the probabilities are not all the same, as is the case with the sums of pairs of dice, it gets more complicated.

One way to solve this problem to use Markov chains. This is conceptually straightforward, but a bit computationally elaborate. As we roll the pair of dice, we consider ourselves in a state represented by an 11-dimensional vector $\langle a_2, a_3, \cdots, a_{12}\rangle$, where $a_i = 0$ or 1 for $i = 2, \cdots 12$, and $a_i = 1$ if and only if a sum of i has been achieved thus far. We start in the state $\langle 0, 0, \cdots, 0\rangle$ and we want to know the expected number of rolls until we are in the state $\langle 1, 1, \cdots, 1\rangle$ (this is our one absorbing state).

From each state except the absorbing state, the probability of moving to another state is determined by the probability of rolling each of the so-far unrolled sums.

We can create the transition matrix M for this process, and use it to calculate the expected value (see Appendix D for more details on this method). The basic idea is to create a transition matrix M, and from that take a sub-matrix Q, from which the matrix $N = (I-Q)^{-1}$ is calculated. Then the expected number of rolls needed will be the sum of the values in the first row of N.

Below is some GP code that creates the transition matrix and calculates the expected value.

A helpful idea here is to assign each state a positive integer via the map

$$s(\langle a_2, a_2, \cdots, a_12\rangle) = \sum_{i=2}^{12} 2^{i-2}a_i$$

so we can represent each state as a positive integer, rather than as a vector.

```
A=vector(11);
s(A)=sum(i=1,11,2^(i-1)*A[i]);
p=sum(i=1,6,x^i)*1/6;
B=vector(12);
for(i=2,12,B[i]=polcoeff(p^2,i,x));
M=matrix(2048,2048); \\ allocatemem() as needed to get the required memory
for(a2=0,1,for(a3=0,1,for(a4=0,1,for(a5=0,1,for(a6=0,1,\
for(a7=0,1,for(a8=0,1,for(a9=0,1,for(a10=0,1,for(a11=0,1,for(a12=0,1,\
A=[a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12];\
s0=s(A);\
for(j=2,12,C=A;if(A[j-1]<1,C[j-1]=1;M[s0+1,s(C)+1]=B[j];));\
M[s0+1,s0+1] = sum(j=2,12,B[j]*A[j-1]);\
)))))))))))
print("Found M.");
Q=matrix(2047,2047);
print("Now we have Q.");
for(i=1,2047,for(j=1,2047,Q[i,j]=M[i,j]));
N=(matid(2047)-Q)^(-1)
print("Now we have N.");
print1("The expected value is ",sum(j=1,2047,N[1,j]));
```

The expected number of rolls is

$$\frac{769767316159}{12574325400} = 61.217384763957....$$

In [2] (equation 14b) , the authors give the following formula for calculating such an expected value:

$$E = \sum_{q=0}^{m-1} (-1)^{m-1-q} \sum_{|J|=q} \frac{1}{1-P_J}$$

where m is the number of "coupons", J is a subset of all possible coupons, and $P_J = \sum_{j \in J} p_j$ where p_j is the probability of drawing the j-th coupon on any turn.

This requires, essentially, summing over the power set of the set of all coupons. Here is some PARI/GP code that finds the expected value using this formula:

```
A=vector(11);
p=sum(i=1,6,x^i)*1/6;
B=vector(12);
for(i=2,12,B[i]=polcoeff(p^2,i,x));
E=0;
for(a2=0,1,for(a3=0,1,for(a4=0,1,for(a5=0,1,for(a6=0,1,\
for(a7=0,1,for(a8=0,1,for(a9=0,1,for(a10=0,1,for(a11=0,1,for(a12=0,1,\
A=[a2,a3,a4,a5,a6,a7,a8,a9,a10,a11,a12];\
q=sum(i=1,11,A[i]);\
P=sum(i=1,11,A[i]*B[i+1]);\
if(P<1,E=E+(-1)^(11-1-q)*1/(1-P));\
)))))))))));
print(E);
```

This is definitely a more efficient way to compute this: on one machine, this code takes about $1/10000$ as much time as the Markov chain method.

With a small modification to the above code, we can find the expected number of rolls until all sixteen possible sums are attained when rolling three dice. The expected value is

$$\frac{3278040865018061671709856608149454942231705937716894332690966681078219322742794124 3843}{9685362624925258411110963697862669592736686405619920066104594539510791426924597160 0}$$

which is about 338.45308554395589, and here is the PARI/GP code:

```
A=vector(16);
p=sum(i=1,6,x^i)*1/6;
B=vector(18);
for(i=3,18,B[i]=polcoeff(p^3,i,x));
E=0;
for(a3=0,1,for(a4=0,1,for(a5=0,1,for(a6=0,1,\
for(a7=0,1,for(a8=0,1,for(a9=0,1,for(a10=0,1,for(a11=0,1,for(a12=0,1,\
for(a13=0,1,for(a14=0,1,for(a15=0,1,for(a16=0,1,for(a17=0,1,for(a18=0,1,\
A=[a3,a4,a5,a6,a7,a8,a9,a10,a11,a12,a13,a14,a15,a16,a17,a18];\
q=sum(i=1,16,A[i]);\
P=sum(i=1,16,A[i]*B[i+2]);\
if(P<1,E=E+(-1)^(16-1-q)*1/(1-P));\
))))))))))))))));
print(E);
```

29. *A die is rolled repeatedly and summed. What can you say about the expected number of rolls until the sum is greater than or equal to n?*

When a six-sided die is rolled, the expected value of the roll is $\frac{7}{2}$.

So, to reach a sum of n, we expect to need about $\dfrac{2}{7}n = \dfrac{n}{3.5}$ rolls.

That's very rough. Let's find a more precise expression of the expected value.

Define E_n to be the expected number of rolls until the sum is at least n.

Then $E_0 = 0$ and $E_1 = 1$.

Since on the first roll, there is a $5/6$ probability of getting at least 2, and, if we don't, the sum will be at least 2 after the second roll, we have

$$E_2 = 1 + \frac{1}{6} = \frac{7}{6}.$$

That is, $E_2 = 1 + \frac{1}{6}E_1$.

Suppose $n = 3$. On the first roll, we will get 3 or greater with probability $4/6$. If we get a 1, then the expected number of additional rolls is E_2, and if get a 2, then the expected number of additional rolls is E_1. Hence,

$$E_3 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 = \frac{49}{36}.$$

In the same way,

$$E_4 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 = \frac{343}{216}$$
$$E_5 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{1}{6}E_4 = \frac{2401}{1296}$$
$$E_6 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{1}{6}E_4 + \frac{1}{6}E_5 = \frac{16807}{7776}.$$

Suppose $n > 6$. In order to reach a sum of at least n, the final roll must be i after achieving a sum of at least $n - i$. Hence,

$$E_n = 1 + \frac{1}{6}\sum_{i=1}^{6} E_{n-i}. \tag{3.13}$$

To help us study E_n, we will find a generating function for E_n. Define

$$f(x) = \sum_{n=0}^{\infty} E_n x^n.$$

We wish to find a simple, closed-form expression for f.

Multiplying equation 3.13 by x^i and summing from $n = 6$ to ∞ we have

$$\sum_{n=6}^{\infty} E_n x^n = \sum_{n=6}^{\infty}\left(x^n + \frac{1}{6}\sum_{i=1}^{6} E_{n-i}x^n\right). \tag{3.14}$$

Define $f_k(x) = \sum_{i=0}^{k} E_i x^i$. Then equation 3.14 yields

$$\begin{aligned}
f(x) - f_5(x) = & \\
&= \sum_{n=6}^{\infty} x^n + \frac{1}{6}\sum_{i=1}^{6}\left(\sum_{n=6}^{\infty} E_{n-i}x^n\right)\\
&= \frac{x^6}{1-x} + \frac{1}{6}\sum_{n=6}^{\infty} E_{n-1}x^n + \frac{1}{6}\sum_{n=6}^{\infty} E_{n-2}x^n + \cdots + \frac{1}{6}\sum_{n=6}^{\infty} E_{n-5}x^n + \frac{1}{6}\sum_{n=6}^{\infty} E_{n-6}x^n\\
&= \frac{x^6}{1-x} + \frac{1}{6}\sum_{n=5}^{\infty} E_n x^{n+1} + \frac{1}{6}\sum_{n=4}^{\infty} E_n x^{n+2} + \cdots + \frac{1}{6}\sum_{n=1}^{\infty} E_n x^{n+5} + \frac{1}{6}\sum_{n=0}^{\infty} E_n x^{n+6}\\
&= \frac{x^6}{1-x} + \frac{1}{6}x\sum_{n=5}^{\infty} E_n x^n + \frac{1}{6}x^2\sum_{n=4}^{\infty} E_n x^n + \cdots + \frac{1}{6}x^5\sum_{n=1}^{\infty} E_n x^n + \frac{1}{6}x^6\sum_{n=0}^{\infty} E_n x^n\\
&= \frac{x^6}{1-x} + \frac{1}{6}x(f(x) - f_4(x)) + \frac{1}{6}x^2(f(x) - f_3(x)) + \cdots + \frac{1}{6}x^5(f(x) - f_0(x)) + \frac{1}{6}x^6 f(x)
\end{aligned}$$

Using the fact that $f_0(x) = 0$ and simplifying, we have

$$f(x) - f_5(x) = \frac{x^6}{1-x} + \frac{1}{6}(x + x^2 + \cdots + x^6)f(x) - \frac{1}{6}(xf_4(x) + x^2 f_3(x) + x^3 f_2(x) + x^4 f_1(x)).$$

Since the f_i are just polynomials, we can solve this for $f(x)$ and find that

$$f(x) = \frac{6x}{x^7 - 7x + 6} = \frac{6x}{(x-1)^2(x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6)}.$$

Applying the method of partial fractions, we find

$$f(x) = \frac{2/7}{(x-1)^2} - \frac{4/21}{x-1} + \frac{\frac{2}{21}(2x^4 + 3x^3 - 10x - 30)}{x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6}$$

$$= \frac{2}{7}\sum_{n=0}^{\infty}(n+1)x^n + \frac{4}{21}\sum_{n=0}^{\infty}x^n + Z(x)$$
$$= \sum_{n=0}^{\infty}\left(\frac{2}{7}n + \frac{10}{21}\right)x^n + Z(x), \text{ say.}$$

The poles of $Z(x)$ are approximately

$$-1.491797988139901,$$
$$-0.805786469389031 \pm 1.222904713374410i,$$
$$\text{and } 0.551685463458982 \pm 1.253348860277206i.$$

The minimum modulus among these poles is $R = 1.3693941054897684$. As a result, the n-th coefficient a_n of the power series of $Z(x)$ satisfies

$$a_n = O\left(\left(\frac{1}{R}\right)^n\right) = O(0.73024997^n).$$

(See, for example, [3], (June 26, 2009 edition), Theorem IV.7 (Exponential Growth Formula), page 244)).

Thus, we may conclude that

$$E_n = \frac{2}{7}n + \frac{10}{21} + O(0.73024997^n).$$

So, E_n is very closely approximated by $\frac{2}{7}n + \frac{10}{21}$ for all but the smallest values of n.

Here is a short table of values. Let $g(n) = \frac{2}{7}n + \frac{10}{21}$.

n	E_n	$g(n)$	$E_n - g(n)$
1	$1 \approx 1.000000000$	0.7619047619	0.2380952381
2	$7/6 \approx 1.166666667$	1.047619048	0.1190476190
3	$49/36 \approx 1.361111111$	1.333333333	0.02777777778
4	$343/216 \approx 1.587962963$	1.619047619	−0.03108465608
5	$2401/1296 \approx 1.852623457$	1.904761905	−0.05213844797
6	$16807/7776 \approx 2.161394033$	2.190476190	−0.02908215755
7	$117649/46656 \approx 2.521626372$	2.476190476	0.04543589555
8	$776887/279936 \approx 2.775230767$	2.761904762	0.01332600513
9	$5111617/1679616 \approx 3.043324784$	3.047619048	−0.004294263859
10	$33495175/10077696 \approx 3.323693729$	3.333333333	−0.009639604132
15	4.760008377	4.761904762	−0.001896385320
20	6.190195199	6.190476190	−0.0002809914888
25	7.619081612	7.619047619	0.00003399272215
30	9.047634594	9.047619048	0.00001554676498
35	10.47619480	10.47619048	0.000004327490791
40	11.90476290	11.90476190	0.0000009934675945
45	13.33333351	13.33333333	0.0000001795084385
50	14.76190478	14.76190476	0.00000001842084560
60	17.61904762	17.61904762	−0.000000000990023724
70	20.47619048	20.47619048	$-8.20317511 \times 10^{-11}$
80	23.33333333	23.33333333	$-1.896370261 \times 10^{-12}$
90	26.19047619	26.19047619	$6.32378328 \times 10^{-14}$
100	29.04761905	29.04761905	$6.478729760 \times 10^{-15}$

We see that E_n is extremely closely approximated by $\frac{2}{7}n + \frac{10}{21}$ as n gets large.

30. *A die is rolled repeatedly and summed. Show that the expected number of rolls until the sum is a multiple of n is n.*

We will treat small n first. We will use the fact that the expected number of additional rolls until the sum is a multiple of n depends only on the residue class of the sum modulo n (e.g., the expected number of additional rolls until the sum is a multiple of 4 is the same whether the sum is 3, 7, 11, or any other value congruent to 3 modulo 4).

Let $n = 2$. Then the expected number of rolls, E until the sum is a multiple of n is

$$E = 1 + \frac{1}{2}E_1$$

and

$$E_1 = 1 + \frac{1}{2}E_1$$

where E_1 is the expected number of additional rolls from an odd sum (i.e., a sum congruent to 1 mod 2). These equations arise from the fact that half of the values $\{1,2,3,4,5,6\}$ are even and half are odd, so there is a one-half chance of the first roll ending in an odd sum, and from there, a one-half chance of staying with an odd sum. These two equations easily lead to $E = 2$.

Let $n = 3$, and let E be the expected number of rolls until the sum is a multiple of three, and E_1 and E_2 be the expected number of additional rolls from a sum congruent to 1 or 2 mod 3. Then we have

$$\begin{aligned} E &= 1 + \frac{1}{3}E_1 + \frac{1}{3}E_2 \\ E_1 &= 1 + \frac{1}{3}E_1 + \frac{1}{3}E_2 \\ E_2 &= 1 + \frac{1}{3}E_1 + \frac{1}{3}E_2 \end{aligned}$$

As in the $n = 2$ case, the fact that $\{1,2,3,4,5,6\}$ are uniformly distributed mod 3 results in three identical expressions, and so we find $E = E_1 = E_2 = 3$.

The $n = 4$ case is a little different, since $\{1,2,3,4,5,6\}$ is not uniformly distributed modulo 4. As a result, our equations, following the scheme above, are

$$\begin{aligned} E &= 1 + \frac{1}{3}E_1 + \frac{1}{3}E_2 + \frac{1}{6}E_3 \\ E_1 &= 1 + \frac{1}{6}E_1 + \frac{1}{3}E_2 + \frac{1}{3}E_3 \\ E_2 &= 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{3}E_3 \\ E_3 &= 1 + \frac{1}{3}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 \end{aligned}$$

Solving this system, we find $E_1 = \frac{98}{25}, E_2 = \frac{84}{25}, E_3 = \frac{86}{25}$, and $E = 4$.

The $n = 5$ case is similar. We have

$$\begin{aligned} E &= 1 + \frac{1}{3}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{1}{6}E_4 \\ E_1 &= 1 + \frac{1}{6}E_1 + \frac{1}{3}E_2 + \frac{1}{6}E_3 + \frac{1}{6}E_4 \\ E_2 &= 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{3}E_3 + \frac{1}{6}E_4 \end{aligned}$$

$$E_3 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{1}{3}E_4$$
$$E_4 = 1 + \frac{1}{6}E_1 + \frac{1}{6}E_2 + \frac{1}{6}E_3 + \frac{1}{6}E_4$$

which yields $E_1 = \frac{1554}{311}, E_2 = \frac{1548}{311}, E_3 = \frac{1512}{311}, E_4 = \frac{1296}{311}$ and $E = 5$.

For $n = 6$, we can conclude more easily. Since $\{1, 2, 3, 4, 5, 6\}$ is uniformly distributed modulo 6, we have $E = E_1 = E_2 = E_3 = E_4 = E_5$ and so

$$E = 1 + \frac{5}{6}E$$

and thus $E = 6$.

Suppose $n > 6$. Define E and E_i as above. Then we have the following system of equations.

$$\begin{aligned}
E &= 1 + \frac{1}{6}E_1 + \cdots + \frac{1}{6}E_6 \\
E_1 &= 1 + \frac{1}{6}E_2 + \cdots + \frac{1}{6}E_7 \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
E_{n-7} &= 1 + \frac{1}{6}E_{n-6} + \cdots + \frac{1}{6}E_{n-1} \\
E_{n-6} &= 1 + \frac{1}{6}E_{n-5} + \cdots + \frac{1}{6}E_{n-1} \\
E_{n-5} &= 1 + \frac{1}{6}E_{n-4} + \cdots + \frac{1}{6}E_{n-1} + \frac{1}{6}E_1 \\
E_{n-4} &= 1 + \frac{1}{6}E_{n-3} + \cdots + \frac{1}{6}E_{n-1} + \frac{1}{6}E_1 + \frac{1}{6}E_2 \\
E_{n-3} &= 1 + \frac{1}{6}E_{n-2} + \cdots + \frac{1}{6}E_{n-1} + \frac{1}{6}E_1 + \cdots \frac{1}{6}E_3 \\
E_{n-2} &= 1 + \frac{1}{6}E_{n-1} + \cdots + \frac{1}{6}E_{n-1} + \frac{1}{6}E_1 + \cdots \frac{1}{6}E_4 \\
E_{n-1} &= 1 + \frac{1}{6}E_1 + \cdots + \frac{1}{6}E_5
\end{aligned}$$

Summing, and counting, we have

$$E + \sum_{i=1}^{n-1} E_i = n + \sum_{i=1}^{n-1} 6 \cdot \frac{1}{6}E_i = n + \sum_{i=1}^{n-1} E_i$$

and so $E = n$.

A curious feature of this is that a uniform distribution of dice values is actually not necessary to have n rolls be the expected value. A variety of other kinds of die distributions (appear to) yield n also. So a question is: what are necessary conditions on the values of a die so that n is the expected number of rolls until the sum is a multiple of n?

31. *A fair, n-sided die is rolled and summed until the sum is at least n. What is the expected number of rolls?*

To solve this, we will use some recursive expressions.

Let $E(m)$ be the expected number of rolls until the sum is at least n, starting with a sum of m.

Then we have:

$$\begin{aligned}
E(n) &= 0 \\
E(n-1) &= 1 \\
E(n-2) &= 1 + \frac{1}{n}E(n-1) = 1 + \frac{1}{n} \\
E(n-3) &= 1 + \frac{1}{n}E(n-2) + \frac{1}{n}E(n-1) \\
&= 1 + \frac{1}{n}(1 + \frac{1}{n}E(n-1)) + \frac{1}{n} \\
&= 1 + \frac{2}{n} + \frac{1}{n^2} \\
E(n-4) &= 1 + \frac{1}{n}E(n-3) + \frac{1}{n}E(n-2) + \frac{1}{n}E(n-1) \\
&= 1 + \frac{3}{n} + \frac{3}{n^2} + \frac{1}{n^3}
\end{aligned}$$

Suppose

$$E(n-k) = \sum_{i=0}^{k-1} \frac{\binom{k-1}{i}}{n^i}$$

for $1 \leq k < r$.

Then

$$\begin{aligned}
E(n-r) &= 1 + \frac{1}{n}\sum_{m=1}^{r-1} E(n-m) = 1 + \frac{1}{n}\sum_{m=1}^{r-1}\sum_{i=0}^{m-1}\frac{\binom{m-1}{i}}{n^i} \\
&= 1 + \frac{1}{n}\sum_{i=0}^{r-2}\frac{1}{n^i}\sum_{j=i}^{r-2}\binom{j}{i} = 1 + \frac{1}{n}\sum_{i=0}^{r-2}\frac{1}{n^i}\binom{r-1}{i+1} \\
&= 1 + \sum_{i=0}^{r-2}\frac{1}{n^{i+1}}\binom{r-1}{i+1} = 1 + \sum_{i=1}^{r-1}\frac{1}{n^i}\binom{r-1}{i} \\
&= \sum_{i=0}^{r-1}\frac{1}{n^i}\binom{r-1}{i}
\end{aligned}$$

(Here we have used the lovely identity $\displaystyle\sum_{k=r}^{m}\binom{k}{r} = \binom{m+1}{r+1}$.)

Thus by induction, we have $E(n-k) = \displaystyle\sum_{i=0}^{k-1}\frac{\binom{k-1}{i}}{n^i}$ for $1 \leq k \leq n$.

The value we seek is $E(0)$:

$$E(0) = \sum_{i=0}^{n-1}\frac{\binom{n-1}{i}}{n^i} = \left(1 + \frac{1}{n}\right)^{n-1}.$$

We observe the curiosity that as $n \to \infty$, $E(0) \to e$.

We have the following values:

n	$E(0)$ exact	$E(0)$ approx.
1	1	1.0000
2	3/2	1.5000
3	16/9	1.7778
4	125/64	1.9531
5	1296/625	2.0736
6	16807/7776	2.1614
7	262144/117649	2.2282
8	4782969/2097152	2.2807
9	100000000/43046721	2.3231
10	2357947691/1000000000	2.3579
20		2.5270
100		2.6780
500		2.7101
1000		2.7142

32. *A die is rolled once; call the result N. Then N dice are rolled once and summed. What is the distribution of the sum? What is the expected value of the sum? What is the most likely value?*

What the heck, take it one more step: roll a die; call the result N. Roll N dice once and sum them; call the result M. Roll M dice once and sum. What's the distribution of the sum, expected value, most likely value?

Since each of the possible values $\{1, 2, 3, 4, 5, 6\}$ of N are equally likely, we can calculate the distribution by summing the individual distributions of the sum of 1, 2, 3, 4, 5, and 6 dice, each weighted by $\frac{1}{6}$. We can do this using polynomial generating functions. Let

$$p = \frac{1}{6}(x + x^2 + x^3 + x^4 + x^5 + x^6).$$

Then the distribution of the sum is given by the coefficients of the polynomial

$$D = \sum_{i=1}^{6} \frac{1}{6} p^i$$

$$= \frac{1}{279936}x^{36} + \frac{1}{46656}x^{35} + \frac{7}{93312}x^{34} + \frac{7}{34992}x^{33} + \frac{7}{15552}x^{32} + \frac{7}{7776}x^{31} + \frac{77}{46656}x^{30} + \frac{131}{46656}x^{29} + \frac{139}{31104}x^{28} + \frac{469}{69984}x^{27} + \frac{889}{93312}x^{26} + \frac{301}{23328}x^{25} + \frac{4697}{279936}x^{24} + \frac{245}{11664}x^{23} + \frac{263}{10368}x^{22} + \frac{691}{23328}x^{21} + \frac{1043}{31104}x^{20} + \frac{287}{7776}x^{19} + \frac{11207}{279936}x^{18} + \frac{497}{11664}x^{17} + \frac{4151}{93312}x^{16} + \frac{3193}{69984}x^{15} + \frac{1433}{31104}x^{14} + \frac{119}{2592}x^{13} + \frac{749}{15552}x^{12} + \frac{2275}{46656}x^{11} + \frac{749}{15552}x^{10} + \frac{3269}{69984}x^{9} + \frac{4169}{93312}x^{8} + \frac{493}{11664}x^{7} + \frac{16807}{279936}x^{6} + \frac{2401}{46656}x^{5} + \frac{343}{7776}x^{4} + \frac{49}{1296}x^{3} + \frac{7}{216}x^{2} + \frac{1}{36}x.$$

To get the expected value E, we must calculate

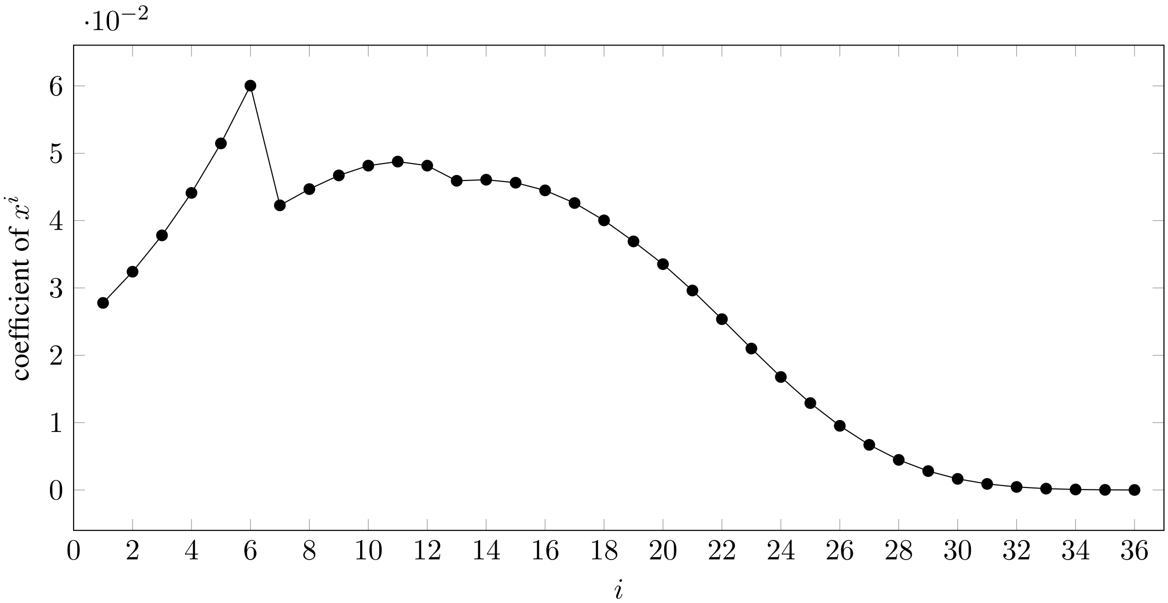
$$E = \sum_{i=1}^{36} i d_i$$

where $D = \sum_{i=1}^{36} d_i x^i$. This works out to $E = \frac{49}{4} = 12.25$.

More simply, one can calculate the expected value of the sum as follows, using the fact that the expected value of a single roll is 3.5:

$$E = \frac{1}{6}\left(3.5 + 2 \times 3.5 + 3 \times 3.5 + \cdots + 6 \times 3.5\right) = 12.25.$$

Since $\sum_{i=1}^{11} d_i = \frac{1255}{2592} < \frac{1}{2}$, and $\sum_{i=1}^{12} d_i = \frac{8279}{15552} > \frac{1}{2}$, we can say that the median value is between 11 and 12.



You can see from the plot of the coefficients of D that 6 is the most likely value. It is perhaps a bit surprising that there are three "local maxima" in the plot, at $i = 6, 11$, and 14.

Okay, now lets do one more step.

After rolling the dice, getting a sum of N, and then rolling N dice, the sum distribution is

$$D_1 = \sum_{i=1}^{6} \frac{1}{6} p^i$$

as above. The coefficient of x^i in D then gives us the probability that the sum of i. Hence if we call the sum M and then roll M dice once, the sum distribution is given by

$$D_2 = \sum_{i=1}^{36} D_1(i) p^i$$

where $D_1(i)$ is the coefficient on x^i in D_1.

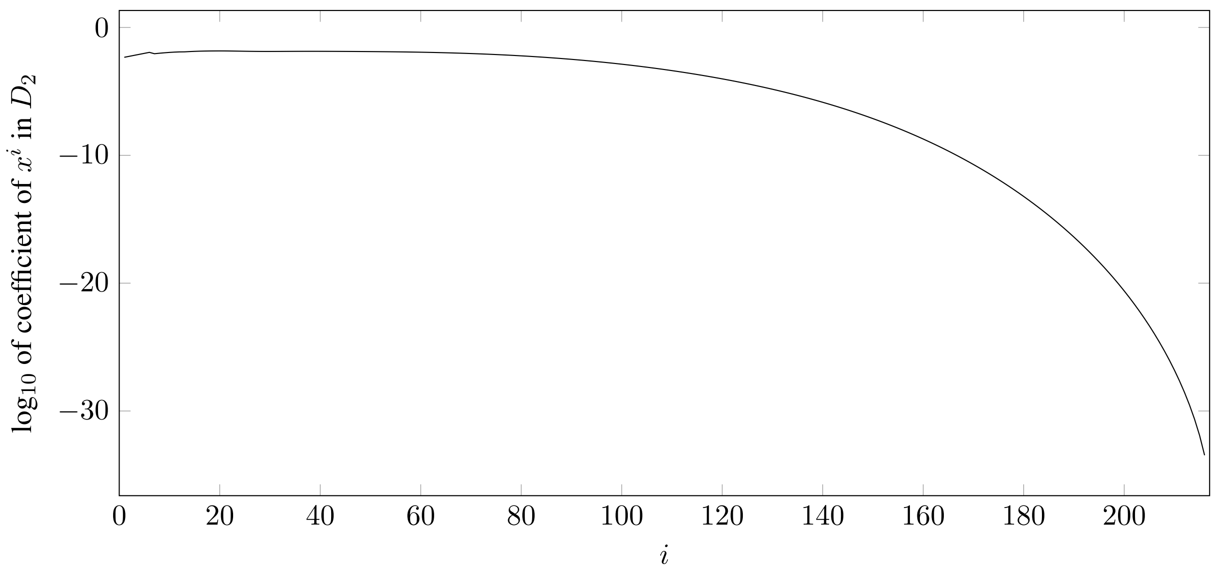
Now, D_2 is a degree 216 polynomial with massive rational coefficients, so there is little point in printing it here. Let $D_2(i)$ be the coefficient on x^i in D_2.

We can find the expected value of the sum as

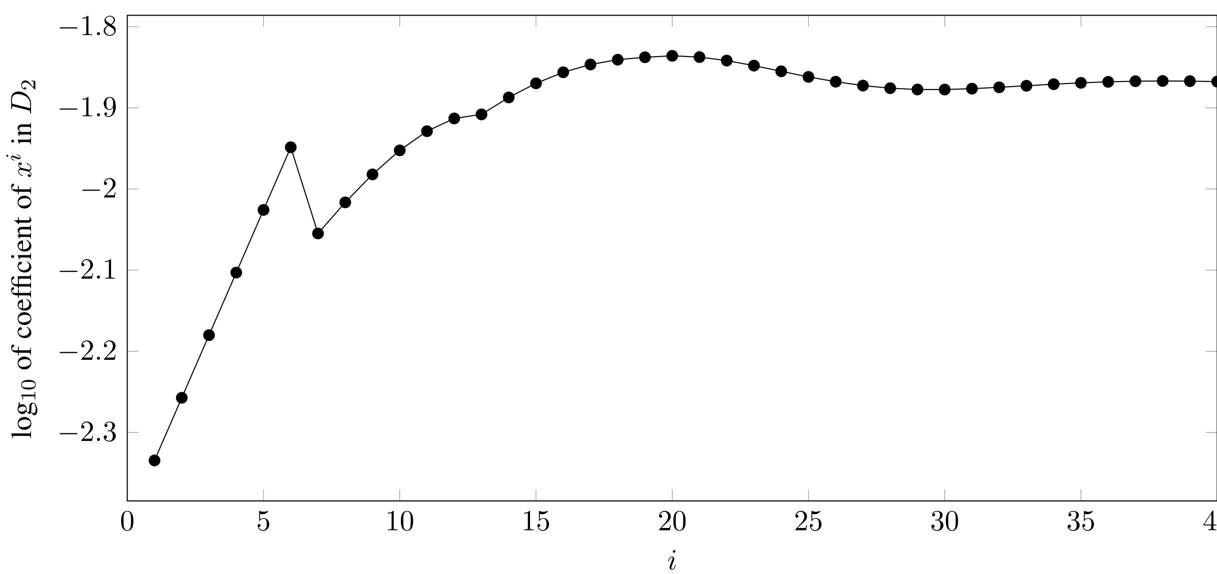
$$\sum_{i=1}^{216} i D_2(i) = \frac{343}{8} = 42.875.$$

Since $\sum_{i=1}^{40} D_2(i) < \frac{1}{2}$, and $\sum_{i=1}^{41} D_2(i) > \frac{1}{2}$, we can say that the median sum is between 40 and 41.

Here's a plot of the distribution:



Here's a plot showing just the coefficients of x^i for small values of i. There are local maxima at $i = 6$, $i = 20$ (the absolute max), and $i = 38$, and a local minimum at $i = 7$ and $i = 30$.



33. *A die is rolled and summed repeatedly. What is the probability that the sum will ever be a given value x? What is the limit of this probability as $x \to \infty$?*

Let's start by considering 2-sided dice, with sides numbered 1 and 2. Let $p(x)$ be the probability that the sum will ever be x. Then $p(1) = 1/2$ since the only way to ever have a sum of 1 is to roll 1 on the first roll. We then have $p(2) = 1/2 + 1/2p(1) = 3/4$, since there are two mutually exclusive ways to get a sum of 2: roll 2 on the first roll, or roll a 1 followed by a 1 on the second roll. Now, extending this idea, we have, for $x > 2$,

$$p(x) = \frac{1}{2}p(x-1) + \frac{1}{2}p(x-2). \tag{3.15}$$

This equation could be used to calculate $p(x)$ for any given value of x. However, this would require calculating p for all lower values. Can we get an *explicit* expression for $p(x)$?

Equation 3.15 is an example of a *linear recurrence relation*. One way to get a solution, or explicit formula, for such a relation is by examining the *auxiliary* equation for equation 3.15:

$$x^2 = \frac{1}{2}x + \frac{1}{2}$$

or

$$x^2 - \frac{1}{2}x - \frac{1}{2} = 0$$

The roots of this equation are

$$\alpha = 1 \text{ and } \beta = -\frac{1}{2}$$

A powerful theorem (see Appendix E) says that

$$p(n) = A\alpha^n + B\beta^n = A + B\left(-\frac{1}{2}\right)^n$$

for constants A and B. Since $p(1) = 1/2$ and $p(2) = 3/4$ we can solve for A and B to find that

$$p(n) = \frac{2}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^n.$$

For 3-sided dice, we have

$$p(1) = \frac{1}{3}, p(2) = \frac{4}{9}, \text{ and } p(3) = \frac{16}{27}$$

with, for $n > 3$,

$$p(n) = \frac{1}{3}\left(p(n-1) + p(n-2) + p(n-3)\right) = \frac{1}{3}\sum_{i=1}^{3} p(n-i).$$

The characteristic equation for this recurrence equation can be written

$$3x^3 - x^2 - x - 1 = 0$$

which has roots

$$\alpha = 1, \beta = -\frac{1}{3} - \frac{\sqrt{2}}{3}i, \text{ and } \gamma = -\frac{1}{3} + \frac{\sqrt{2}}{3}i.$$

Using these, and the fact that

$$p(1) = \frac{1}{3}, p(2) = \frac{4}{9}, \text{ and } p(3) = \frac{16}{27},$$

we find

$$p(n) = \frac{1}{2} + \frac{1}{4}\beta^n + \frac{1}{4}\gamma^n.$$

Since β and γ are complex conjugates, and, in any case, $p(n)$ is always real, we might prefer to write $p(n)$ like this:

$$p(n) = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)^n \cos\left(n\left(\frac{\pi}{2} + \tan^{-1}\frac{1}{\sqrt{2}}\right)\right)$$

Using this formula to generate a table, we see that while $p(n)$ is asymptotic to the value 1/2, it wobbles quite a bit:

x	$p(x)$	$p(x) - p(x-1)$
1	0.333333333333333333333333333333	
2	0.444444444444444444444444444444	0.111111111111111111111111111111
3	0.592592592592592592592592592592 5	0.148148148148148148148148148148 1
4	0.456790123456790123456790123 4	-0.135802469135802469135802469 1
5	0.497942386831275720164609053 4	0.041152263374485596707818930 02
6	0.515775034293552812071330589 8	0.017832647462277091906721536 36
7	0.490169181527206218564243255 6	-0.025605852766346593507087334 24
8	0.501295534217344916933394299 6	0.011126352690138698369151044 04
9	0.502413250012701315856322715 0	0.001117715795356398922928415 384
10	0.497959321919084150451320090 1	-0.004453928093617165405002624 938
11	0.500556035383043461080345701 5	0.002596713463959310629025611 496
12	0.500309535771609642462662835 5	-0.000246499611433818617682866 0183
13	0.499608297691245751331442875 7	-0.000701238080363891131219959 8200
14	0.500157956281966284958150470 9	0.000549658590720533626707595 2194
15	0.500025263248273892917418727 4	-0.000132693033692392040731743 5396
16	0.499930505740495309735670691 3	-0.000094757507778583181748036 04667
17	0.500037908423578495870413296 6	0.000107402683083186134742605 2109
18	0.499997892470782566174500905 1	-0.000040015952795929695912391 45842
19	0.499988768878285457260194964 3	-0.000009123592497108914305940 764722
20	0.500008189924215506435036388 7	0.000019421045930049174841424 32929

Let's skip over 4- and 5-sided dice to deal with 6-sided dice. Let $p(x)$ be the probability that the sum will ever be x. We know that:

$$p(1) = \frac{1}{6}$$

$$p(2) = \frac{1}{6} + \frac{1}{6}p(1) = \frac{7}{36}$$

$$p(3) = \frac{1}{6} + \frac{1}{6}p(2) + \frac{1}{6}p(1) = \frac{49}{216}$$

$$p(4) = \frac{1}{6} + \frac{1}{6}p(3) + \frac{1}{6}p(2) + \frac{1}{6}p(1) = \frac{343}{1296}$$

$$p(5) = \frac{1}{6} + \frac{1}{6}p(4) + \frac{1}{6}p(3) + \frac{1}{6}p(2) + \frac{1}{6}p(1) = \frac{2401}{7776}$$

$$p(6) = \frac{1}{6} + \frac{1}{6}p(5) + \frac{1}{6}p(4) + \frac{1}{6}p(3) + \frac{1}{6}p(2) + \frac{1}{6}p(1) = \frac{16807}{46656}$$

and for $x > 6$,

$$p(x) = \frac{1}{6}\sum_{i=1}^{6} p(x-i).$$

The corresponding characteristic equation

$$6x^6 - x^5 - x^4 - x^3 - x^2 - x - 1 = 0$$

has roots, approximately,

$$\begin{aligned}
A_1 &= 1 \\
A_2 &= -0.67033204760309682774 \\
A_3 &= -0.37569519922525992469 - 0.57017516101141226375i \\
A_4 &= \overline{A_3} \\
A_5 &= 0.29419455636014167190 - 0.66836709744330106478i \\
A_6 &= \overline{A_5}
\end{aligned}$$

Solving the system of equations

$$p(j) = \sum_{i=1}^{6} c_i A_i^j, j = 1, \ldots, 6$$

we find $c_1 = \frac{2}{7}$ and $c_i = \frac{1}{7}$ for $i = 2, \ldots, 7$.

Hence, we may express $p(n)$ as

$$p(n) = \frac{2}{7} + \frac{1}{7} \sum_{i=2}^{6} A_i^n.$$

Since all the A_i except A_1 have absolute value less than one, we may conclude that

$$\lim_{n \to \infty} p(n) = \frac{2}{7}.$$

Here's a table of the values of $p(x)$ and $p(x) - p(x-1)$ for $x \leq 20$:

x	$p(x)$	$p(x) - p(x-1)$
1	0.16666666666666666666666666666	
2	0.19444444444444444444444444444	0.02777777777777777777777777777
3	0.22685185185185185185185185185 18518	0.03240740740740740740740740740
4	0.26466049382716049382716049382	0.03780864197530864197530864197
5	0.30877057613168724279835390940	0.04411008230452674897119341563
6	0.36023233882030178326474622770	0.05146176268861454046639231824
7	0.25360439529035208047553726560	-0.10662794352994970278920896200
8	0.26809401672763298277701569880	0.01448962143728090230147843316
9	0.28036894544149773916577757450	0.01227492871386475638876187573
10	0.28928846103977205371809852830	0.00891951559827431455232095378
11	0.29339312224187398036658820070	0.00410466120210192664848967241
12	0.29083021326023843662796058260	-0.00256290898163554373862761811
13	0.27926319233356121218849630840	-0.01156702092667722443946427417
14	0.28353965850742940080732281550	0.00427646617386818861882650713
15	0.28611393213739547047904066830	0.00257427362996606967171785279
16	0.28707142992004509236458451730	0.00095749778264962188554384897
17	0.28670192473342393213899884880	-0.00036950518662116022558566849
18	0.28558672514868225743440062350	-0.00111519958474167470459822531
19	0.28471281046342289423547396370	-0.00087391468525936319892665984
20	0.28562108015173317457663690620	0.00090826968831028034116294254

Here's another proof that $p(n)$ approaches $\frac{2}{7}$ (proof idea from Marc Holtz).

First, let's define a sequence of vectors $v(i)$:

$$v(i) = \langle p(i), p(i-1), p(i-2), p(i-3), p(i-4), p(i-5) \rangle.$$

If we then define the matrix M:

$$M = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Then it's not hard to show that

$$Mv(i) = v(i+1)$$

What we are interested in, then is $M^{\infty} v(j) = \lim_{i \to \infty} v(i)$, where j is any finite value (but we may as well take it to be six, since we've calculated $p(1)$,...,$p(6)$ already

Note that each entry of M is between 0 and 1, each row of M sums to one, and M^6 has no zero entries:

$$M^6 = \begin{pmatrix} \frac{16807}{46656} & \frac{9031}{46656} & \frac{7735}{46656} & \frac{6223}{46656} & \frac{4459}{46656} & \frac{2401}{46656} \\ \frac{2401}{7776} & \frac{2401}{7776} & \frac{1105}{7776} & \frac{889}{7776} & \frac{637}{7776} & \frac{343}{7776} \\ \frac{343}{1296} & \frac{343}{1296} & \frac{343}{1296} & \frac{127}{1296} & \frac{91}{1296} & \frac{49}{1296} \\ \frac{49}{216} & \frac{49}{216} & \frac{49}{216} & \frac{49}{216} & \frac{13}{216} & \frac{7}{216} \\ \frac{7}{36} & \frac{7}{36} & \frac{7}{36} & \frac{7}{36} & \frac{7}{36} & \frac{1}{36} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

So, we can consider M to be a transition matrix of a regular Markov system. Hence M^∞ is a matrix with all identical rows given by the vector w where the sum of the entries of w equals 1, and

$$wM = w.$$

A little simple algebra shows that

$$w = \left\langle \frac{2}{7}, \frac{5}{21}, \frac{4}{21}, \frac{1}{7}, \frac{2}{21}, \frac{1}{21} \right\rangle$$

Hence, $v(\infty)$ is a vector of six identical probabilities equal to

$$w \cdot v(6) = \frac{2}{7}$$

Thus, $\lim\limits_{i\to\infty} = \frac{2}{7}$.

More questions:

(a) Notice that while $p(x)$ is settling down on $\frac{2}{7}$, it does so quite non-monotonically: $p(x)$ increases to its maximum at $x = 6$, and then wobbles around quite a bit. Is the sequence $p(i)$ eventually monotonic, or does it always wobble?

34. *A die is rolled once. Call the result N. Then, the die is rolled N times, and those rolls which are equal to or greater than N are summed (other rolls are not summed). What is the distribution of the resulting sum? What is the expected value of the sum?*

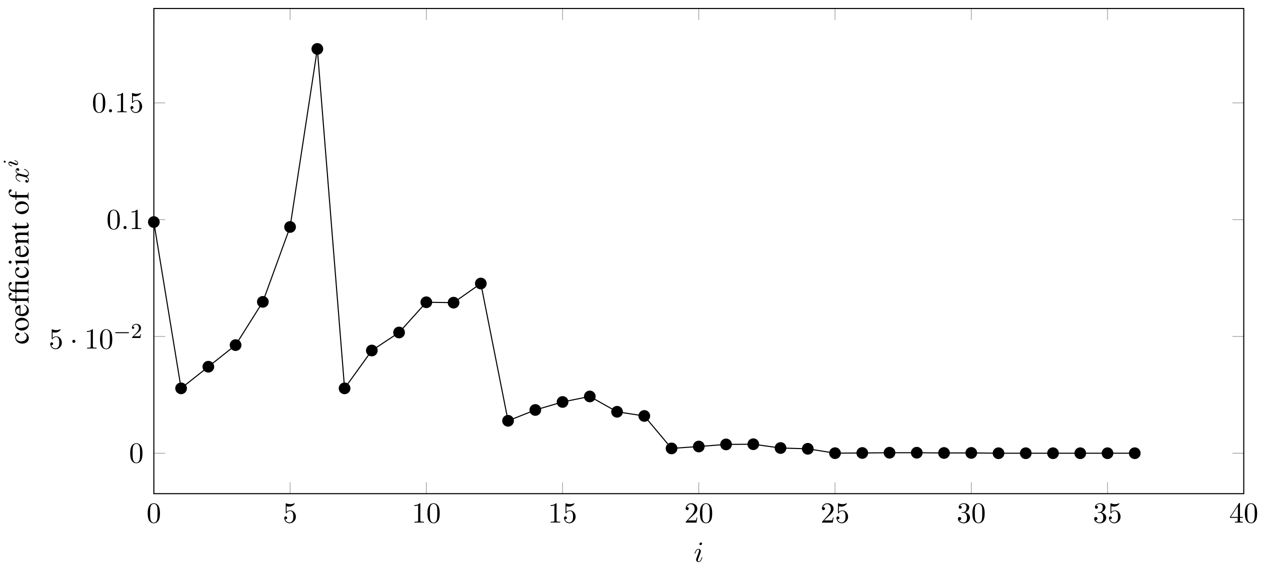
This is a perfect problem for the application of the polynomial representation of the distribution of sums.

The probability of a sum of k is the coefficient on x^k in the polynomial

$$\frac{1}{6}\left(\frac{1}{6}\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)\right) + \frac{1}{6}\left(\frac{1}{6}\left(1 + x^2 + x^3 + x^4 + x^5 + x^6\right)\right)^2 +$$
$$\frac{1}{6}\left(\frac{1}{6}\left(2 + x^3 + x^4 + x^5 + x^6\right)\right)^3 + \frac{1}{6}\left(\frac{1}{6}\left(3 + x^4 + x^5 + x^6\right)\right)^4 +$$
$$\frac{1}{6}\left(\frac{1}{6}\left(4 + x^5 + x^6\right)\right)^5 + \frac{1}{6}\left(\frac{1}{6}\left(5 + x^6\right)\right)^6$$

$$= \frac{1}{279936}x^{36} + \frac{1}{7776}x^{30} + \frac{5}{46656}x^{29} + \frac{5}{23328}x^{28} + \frac{5}{23328}x^{27} + \frac{5}{46656}x^{26} + \frac{1}{46656}x^{25} + \frac{59}{31104}x^{24} + \frac{13}{5832}x^{23} + \frac{5}{1296}x^{22} + \frac{11}{2916}x^{21} + \frac{67}{23328}x^{20} + \frac{1}{486}x^{19} + \frac{1117}{69984}x^{18} + \frac{23}{1296}x^{17} + \frac{7}{288}x^{16} + \frac{16}{729}x^{15} + \frac{1}{54}x^{14} + \frac{1}{72}x^{13} + \frac{6781}{93312}x^{12} + \frac{47}{729}x^{11} + \frac{377}{5832}x^{10} + \frac{67}{1296}x^{9} + \frac{19}{432}x^{8} + \frac{1}{36}x^{7} + \frac{8077}{46656}x^{6} + \frac{565}{5832}x^{5} + \frac{7}{108}x^{4} + \frac{5}{108}x^{3} + \frac{1}{27}x^{2} + \frac{1}{36}x + \frac{27709}{279936}$$

So, that's the distribution. Here's a plot of the distribution:



The expected value is simply the sum of i times the coefficient on x^i in the distribution polynomial. The result is $\frac{133}{18} = 7.38888....$

The probability that the sum is 5 or less is $\dfrac{104077}{279936} = 0.3717...$ while the probability that the sum is 6 or less is $\dfrac{152539}{279936} = 0.5449...$, so we would say the median sum is somewhere between 5 and 6.

35. *Suppose n six-sided dice are rolled and summed. For each six that appears, we sum the six, and reroll that die and sum, and continue to reroll and sum until we roll something other than a six with that die. What is the expected value of the sum? What is the distribution of the sum?*

Each die is independent, so we can work out the distribution for a single die, and get everything we need from that.

To start, we can note that the expected value E when rolling a single die satisfies

$$E = \frac{1}{6}1 + \frac{1}{6}2 + \frac{1}{6}3 + \frac{1}{6}4 + \frac{1}{6}5 + \frac{1}{6}(6 + E)$$

and so $E = 4.2$, so with n dice, the expected sum is $4.2n$.

The first thing we might notice is that the probability of getting a score of $6m+k$, for any non-negative m, and $k \in \{1, 2, 3, 4, 5\}$ is

$$P(6m + k) = \left(\frac{1}{6}\right)^{m+1}$$

Note that it is not possible to score a multiple of 6.

We can create some generating functions (that is, power series where the coefficient on x^m is the probability of getting a final sum of m). For scores congruent to 1 mod 6, we have

$$\frac{1}{6}x + \frac{1}{6^2}x^7 + \frac{1}{6^3}x^{13} + \cdots = \frac{x}{6 - x^6}$$

The generating function for congruence class k mod 6 is x^k times this, so the overall generating function for a single die is

$$\frac{x + x^2 + x^3 + x^4 + x^5}{6 - x^6} = \frac{x^6 - x}{(x-1)(6 - x^6)}$$

So, when rolling r dice, the generating function for the sum is

$$\left(\frac{x^6 - x}{(x-1)(6 - x^6)} \right)^r.$$

Thus, the probability that the sum is k is the coefficient on x^k in the power series representation of the above generating function.

Here's an example of how to calculate with this. Suppose we roll 5 six-sided dice, and want to know the probability that the sum will be greater than 20. We calculate the probability that the sum will be *less than* 20 by truncating the generating function to the 19th degree for a single die and raising it to the fifth power. Let

$$Q = \frac{1}{6}x + \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{6}x^5 + \frac{1}{36}x^7 + \frac{1}{36}x^8 + \frac{1}{36}x^9 + \frac{1}{36}x^{10}$$
$$+ \frac{1}{36}x^{11} + \frac{1}{216}x^{13} + \frac{1}{216}x^{14} + \frac{1}{216}x^{15} + \frac{1}{216}x^{16} + \frac{1}{216}x^{17} + \frac{1}{1296}x^{19}$$

Then, raising Q to the fifth power, we have

$$\frac{1}{7776}x^5 + \frac{5}{7776}x^6 + \frac{5}{2592}x^7 + \frac{35}{7776}x^8 + \frac{35}{3888}x^9 + \frac{121}{7776}x^{10} + \frac{1115}{46656}x^{11} + \frac{1555}{46656}x^{12}$$
$$+ \frac{665}{15552}x^{13} + \frac{2365}{46656}x^{14} + \frac{659}{11664}x^{15} + \frac{2795}{46656}x^{16} + \frac{5695}{93312}x^{17} + \frac{5635}{93312}x^{18} + \frac{5495}{93312}x^{19} + \cdots$$

Summing the coefficients of Q^5 up to x^{19} gives us

$$\frac{44711}{93312},$$

the probability that we score less than 20 when rolling 5 dice. And so, the probability that we score 20 or more is

$$1 - \frac{44711}{93312} = \frac{48601}{93312}.$$

36. *A die is rolled until all sums from* 1 *to* x *are attainable from some subset of rolled faces. For example, if* $x = 3$*, then we might roll until a* 1 *and* 2 *are rolled, or until three* 1*s appear, or until two* 1*s and a* 3*. What is the expected number of rolls?*

I don't have a solution for a general x, but here are some thoughts.

If $x = 1$, then the expected number of rolls is 6.

Let $x = 2$. The expected number of rolls until a 1 or 2 is rolled is 3, and these two outcomes are equally likely. If we rolled a 1, then we need to roll a 1 or a 2, which takes 3 rolls on average. If we rolled a 2, then we must roll a 1, which takes 6 rolls on aveage. Hence, the expected number of rolls is

$$E_2 = 3 + \frac{1}{2}(3) + \frac{1}{2}(6) = \frac{15}{2} = 7.5.$$

We can do the same thing with $x = 3$:

$$E_3 = 2 + \frac{1}{3}(\text{ expected number of rolls after a } 1\text{ }) + \frac{1}{3}(\text{ e.n. of rolls after a } 2\text{ }) + \frac{1}{3}(\text{ e.n. of rolls after a } 3\text{ }).$$

After rolling a 1, we roll until a 1, 2 or 3 appears. If a 1 appears, then we need to roll a 1, 2 or 3. If a 2 appears, we are done. If a 3 appears, then we need to roll a 1 or a 2. Hence the expected number of rolls after rolling a 1 is

$$2 + \frac{1}{3}(2) + \frac{1}{3}(3).$$

After rolling a 2, we roll until a 1 appesrs, which requires 6 rolls on average. After rolling a 3, we still need to achieve subsums 1 and 2, which takes E_2 rolls on average.

Thus,

$$E_3 = 2 + \frac{1}{3}\left(2 + \frac{1}{3}(2) + \frac{1}{3}(3)\right) + \frac{1}{3}6 + \frac{1}{3}E_2 = \frac{139}{18}.$$

We could continue in this way, but instead we can treat the problem via a Markov process.

Create the set of 2^x vectors $V = \{\langle a_1, a_2, \ldots, a_x\rangle : a_i \in \{0,1\}, i = 1, 2, \ldots, x\}$. Each vector corresponds to a state in a Markov chain: if in state $\langle a_1, a_2, \ldots, a_x\rangle$, $a_i = 1$ if and only if a sum of i hss been achieved with the faces rolled so far. The goal is to reach the $\langle 1, 1, \ldots, 1\rangle$ state, which we treat as the absorbing state. The process starts in state $\langle 0, 0, \ldots, 0\rangle$.

For example, with $x = 2$, we have the states and transition matrix

$$\begin{array}{c} \\ \langle 0,0\rangle \\ \langle 1,0\rangle \\ \langle 0,1\rangle \\ \langle 1,1\rangle \end{array}
\begin{array}{c} \begin{array}{cccc} \langle 0,0\rangle & \langle 1,0\rangle & \langle 0,1\rangle & \langle 1,1\rangle \end{array} \\
\left(\begin{array}{cccc} 2/3 & 1/6 & 1/6 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array}$$

For example, the probability of moving from state $\langle 1, 0\rangle$ to state $\langle 1, 1\rangle$ is $1/3$ since, once a 1 has been rolled, we need to roll either a 1 or a 2, hence the probability is $2/6 = 1/3$.

The following PARI/GP code implements this method and outputs the expected number of rolls until all sums $1, 2, \ldots, x$ have been attained. One useful idea is that of converting the state vectors described above into integers by treating the vectors as strings of binary digits. This code applies the method of Appendix D to find the expected value from the transition matrix.

```
x=2; \\ we will roll until all sums 1,2,3,...,x can be achieved from rolled faces
M=matrix(2^x,2^x); \\
A=vector(x); \\
vectonum(V,x) = sum(i=1,x,2^(i-1)*V[i]); \\ gives the state number corresponding to vector V
numtovec(n,x) = B=vector(x);m=n;j=0;while(m>0,B[j+1]=m%2;j=j+1;m=floor(m/2));return(B) \\
{
for(kk=0,2^x-1,
```

```
    for(d=1,6, \\ figure out what state we get to based on each possible face rolled
       BB=numtovec(kk,x); \\ generate the vector for this state
       forstep(r=x,1,-1, \\
          if( (BB[r]>0) && (r+d<=x), BB[r+d]=1)
          ); \\
       if(d<=x,BB[d]=1);
       jj=vectonum(BB,x);\\
       M[kk+1,jj+1] = M[kk+1,jj+1]+1/6;\\ adding one to make indices legal
    ) \\
);
Q=matrix(2^x-1,2^x-1);
for(a=1,2^x-1,for(b=1,2^x-1,Q[a,b]=M[a,b]));
N=(matid(2^x-1)-Q)^(-1);
print(sum(i=1,2^x-1,N[1,i]));
}
```

Using this code, I found the following expected values:

x	expected value(exact)	ev (approx.)	first difference
2	$15/2$	7.5	
3	$138/18$	7.72222	0.222222
4	$9139/1152$	7.93316	0.210937
5	$28669967/3600000$	7.96387	0.030072
6	$777101609/97200000$	7.99487	0.030992
7	$2341848577/291600000$	8.03103	0.036158
8	$883143607/109350000$	8.07630	0.045271
9	$42538515011/5248800000$	8.10442	0.028125
10	$256722093191/31492800000$	8.15177	0.047344
11	$1550818181021/188956800000$	8.20726	0.055492

The first differences are interestingly erratic. I suspect a simple expression for the expected values here is unlikely.

Here's another table. This shows the percentage of the time that all sums 1 through x will have been achieved by a certain number of rolls.

x	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0	8.33	19.9	32.0	43.4	53.3	61.8	68.9	74.8	79.5	83.4	86.5	89.0	91.0
3	0	5.56	15.7	27.9	39.9	50.7	60.0	67.7	74.0	79.0	83.1	86.3	88.9	91.0
4	0	0	9.72	23.5	37.3	49.4	59.3	67.4	73.8	79.0	83.0	86.3	88.9	90.9
5	0	0	8.33	22.5	36.9	49.2	59.3	67.4	73.8	79.0	83.0	86.3	88.9	90.9
6	0	0	5.56	22.2	36.8	49.2	59.3	67.4	73.8	79.0	83.0	86.3	88.9	90.9
7	0	0	2.78	21.5	36.8	49.2	59.3	67.4	73.8	79.0	83.0	86.3	88.9	90.9
8	0	0	0	19.9	36.6	49.2	59.3	67.4	73.8	79.0	83.0	86.3	88.9	90.9
9	0	0	0	17.6	36.1	49.1	59.3	67.4	73.8	79.0	83.0	86.3	88.9	90.9
10	0	0	0	13.9	35.2	49.0	59.3	67.4	73.8	79.0	83.0	86.3	88.9	90.9

This is fascinating. The percentages are nearly constant at many roll numbers, and the variety is almost entirely where the number of rolls is small. Unfortunately, the method used above is too computationally involved to be applied to large x.

It would be nice to get at least a heuristic for the growth of the expected value as a function of x.

Comment: experimentally, if you roll a die 15 times, on average you can create sums from 1 to about 48, whereas if you roll 7 times, on average you can get sums from 1 to about 13.3, and 20 rolls will get you to about 68. 30 rolls will get you to about 105, so it certainly appears that this is approaching 3.5 times the number of rolls.

37. *How long, on average, do we need to roll a die and sum the rolls until the sum is a perfect square* $(1, 4, 9, 16, \dots)$*?*

We can make a very precise estimate of this expected value.

We being by calculating the expectation resulting from rolling the die up to 10^4. That is, if we let E be the value we want, and p_i is the probability that the sum is a square for the first time after i rolls, then

$$E = \sum_{i=1}^{\infty} ip_i = \sum_{i=1}^{10^4} ip_i + \sum_{i=10^4+1}^{\infty} ip_i = E_1 + E_2, \text{ say .}$$

We can calculate E_1 by running through all possible sequences of up to 10000 rolls of the die.

Here is some PARI/GP code that does that:

```
{
mm=62000;
A=vector(mm);
A[1]=1;
B=vector(mm);
p=0;
plast=0;
totProb=0;
lowerE=0;
for(roll=1,10000,
      for(i=1,mm,
         if(A[i]>0,for(d=1,6,
              B[i+d]=B[i+d]+A[i]))
      );
      for(i=1,mm,
           if(issquare(i-1) && B[i]>0,p=p+B[i];B[i]=0));A=B;
           totProb=totProb+(p-plast)*1/6^roll;
           lowerE=lowerE+roll*(p-plast)*1/6^roll;
           plast=p;
           print(roll," ",lowerE*1.," ",1-totProb*1.);
           print();
           B=vector(mm);)
}
```

The result, accurate to more than 50 digits, is

$$E_1 = 7.0797642375511051038955466677464257123892606891396 78$$

At the same time, the calculation tells us that the probability that the sum has not reached a square after 10000 rolls is less than 6.2×10^{-28}.

Hence, $E_2 \leq (6.2^{-28})E'$ where E' is the expected number of rolls needed to reach a square if more than 10000 rolls are needed. Let's get an upper bound on E'.

Let's suppose we are rolling a die and the current sum is $m^2 + 1$ where m is large (say $m > 10$). The next square the sum could hit is $(m + 1)^2$. The probability of hitting this square is between $\frac{1}{6}$ and $\frac{16807}{46656}$, so the probability of missing this square is less than $\frac{5}{6}$, and the expected number of rolls needed to reach that square is certainly less than $(m + 1)^2 - (m^2 + 1)$. If the sum "misses" $(m + 1)^2$, then the next square the sum could hit is $(m + 2)^2$, requiring fewer than $(m + 2)^2 - (m + 1)^2$ additional rolls.

Continuing in this way, we have the upper bound

$$E' < 10000 + \sum_{j=0}^{\infty}((m + 1 + j - m^2)pq^j$$

$$= 10000 + p\left(\sum_{j=0}^{\infty} j^2 q^j + \sum_{j=0}^{\infty}(2m+2)jq^j + \sum_{j=0}^{\infty}(2m+1)q^j\right)$$
$$= 10000 + p\left(\frac{q(1+q)}{(1-q)^3} + \frac{(2m+2)q}{(1-q)^2} + \frac{2m+1}{1-q}\right)$$
$$= 10000 + \frac{16807}{648}m + \frac{184877}{1296}$$
$$< 10000 + 26m + 143$$

where $p = \frac{16807}{46656}$ and $q = \frac{5}{6}$. After 10000 rolls, considering the worst case, we would have $m < 245$ and so $E' < 16513$.

Thus, $E_2 < 10^{-22}$, and so we can conclude that

$$E = 7.079764237551105103895$$

accurate to 21 digits to the right of the decimal.

38. *How long, on average, do we need to roll a die and sum the rolls until the sum is prime? What if we roll until the sum is composite?*

We make a very precise estimate of this first expected value in the following way.

We start by calculating the expectation resulting from rolling the die up to 10^4 times.

That is, if E is the value we want, and p_i is the probability that the sum is prime for the first time after i rolls, then

$$E = \sum_{i=1}^{\infty} ip_i = \sum_{i=1}^{10^4} ip_i + \sum_{i=10^4+1}^{\infty} ip_i = E_1 + E_2, \text{ say .}$$

We can calculate E_1 by running through all possible sequences of up to 10000 rolls of the die. Here is some PARI/GP code that does that:

```
{
mm=62000;
A=vector(mm);
A[1]=1;
B=vector(mm);
p=0;
plast=0;
totProb=0;
lowerE=0;
for(roll=1,10000,
     for(i=1,mm,
        if(A[i]>0,for(d=1,6,
             B[i+d]=B[i+d]+A[i]))
     );
     for(i=1,mm,
         if(isprime(i-1) && B[i]>0,p=p+B[i];B[i]=0));A=B;
         totProb=totProb+(p-plast)*1/6^roll;
         lowerE=lowerE+roll*(p-plast)*1/6^roll;
         plast=p;
         print(roll," ",lowerE*1.," ",1-totProb*1.);
         print();
         B=vector(mm);)
}
```

By setting the numerical precision in GP to display well over 500 decimal places, we can conclude that, to 500 decimal places,

$$
\begin{aligned}
E_1 = 2.&42849791369350423036608190624229927163420183134471\\
&18266468959211216521323257379860460932705658054285\\
&24160047589165194841516565634336164772565943485751\\
&20100473140535884140802682651337652276857652736803\\
&43136681232417851326056596686947409855553312451011\\
&32379770133661680260866153068051346260033855486155\\
&52748670772033743828142893635968820059123417686546\\
&04093838923758726201931868732128985848910810088718\\
&92009240571795609351924253153205397373837440242279\\
&09185701767244213100211303319283551672174728414550
\end{aligned}
$$

Also output by the code above is the probability that, after 10000 rolls, the sum has never been prime. This value is approximately $2.05 \cdot 10^{-552}$. Hence,

$$E = E_1 + (2.06 \cdot 10^{-552})E_{10000}$$

where E_{10000} is the expected number of rolls needed if more than 10000 are needed. Though I do not have a proof, it seems true that E_{10000} is certainly less than 10^{50} (a little experimentation shows that, starting from a sum in the range 10000 to 60000, the expected number of rolls needed is not more than 20. Hence, I suspect the true value of E_{10000} is likely less than 10020, so I think I am making a very safe claim here.)

Thus, we may conclude that, to 500 digits of accuracy, $E = E_1$ as listed above.

If I find a way to prove such an upper bound, I'll be sure to add it.

Now, what if we roll until the sum is composite? This is a much easier question, because we cannot roll indefinitely: there are no prime numbers between the primes 89 and 97, and since $97 - 89 > 6$, if our sum passes 89, it must fall on one of the composites between these two primes. Since 89 is the 24th prime, we must land on a composite on or before the 26th roll.

With a slight modification to the code above, we may run it for 26 rolls and find that the expected value of the number of rolls until the sum is composite is exactly

$$\frac{60513498236803196347}{28430288029929701376} \approx 2.1284869915175750702271582 0.$$

If we let p_n be the probability that the sum will be composite for the first time on the nth roll, then we have the following table.

n	p_n	$\sum_{i=1}^{n} p_i$	$1 - \sum_{i=1}^{n} p_i$
1	0.3333333333333333333	0.3333333333333333333	0.666666666666666666
2	0.3611111111111111111	0.6944444444444444444	0.305555555555555555
3	0.1990740740740740740	0.8935185185185185185 2	0.106481481481481481
4	0.0709876543209876543	0.9645061728395061728 4	0.035493827160493827
5	0.0245627572016460905	0.9890689300411522633 7	0.010931069958847736
6	0.0082304526748971193	0.9972993827160493827 2	0.002700617283950617
7	0.0021612082761774119	0.9994605909922267947 0	0.000539409007773205
8	0.0004441491388507849	0.9999047401310775796 4	0.000095259868922420
9	0.0000773986434994665	0.9999821387745770461 8	0.000017861225422953
10	0.0000138093733594133	0.9999959481479364595 5	0.000004051852063540
11	0.0000031009071914850	0.9999990490551279445 9	0.000000950944872055
12	0.0000007538649927743	0.9999998029201207189 5	0.000000197079879281
13	0.0000001611706083475	0.9999999640907290665 1	0.000000035909270933
14	0.0000000297712612252	0.9999999938619902917 5	0.000000006138009708
15	0.0000000050341888355	0.9999999988961791273 1	0.000000001103820872
16	0.0000000008801500407	0.9999999997763291680 6	2.236708319418220031 E-10
17	1.7735336436591380173 E-10	0.9999999999536825324 2	4.631746757590820138 E-11
18	3.7485246399124685943 E-11	0.9999999999911677788 2	8.832221176783515438 E-12
19	7.3437736466195153487 E-12	0.9999999999985115524 7	1.488447530164000092 E-12
20	1.2781174767468527085 E-12	0.9999999999997896699 5	2.103300534171473825 E-13
21	1.8730953391657100290 E-13	0.9999999999999769794 8	2.302051950057638328 E-14
22	2.1219904608947140874 E-14	0.9999999999999981993 9	1.800614891629241323 E-15
23	1.7081782621714818617 E-15	0.9999999999999999075 6	9.243662945775692717 E-17
24	8.9693076528648355726 E-17	0.9999999999999999972 6	2.743552929109835594 E-18
25	2.7083791736101660356 E-18	0.9999999999999999999 6	3.517375551 E-20
26	3.5173755501430727735 E-20	1	0

We can see that, more than 99 percent of the time, 6 or fewer rolls are needed.

3.3 Non-Standard Dice

39. *Show that the probability of rolling doubles with a non-fair ("fixed") die is greater than with a fair die.*

For a fair, n-sided die, the probability of rolling doubles with it is $n \times \frac{1}{n^2} = \frac{1}{n}$. Suppose we have a "fixed" n-sided die, with probabilities $p_1, ..., p_n$ of rolling sides 1 through n respectively. The probability of rolling doubles with this die is

$$p_1^2 + \cdots + p_n^2.$$

We want to show that this is greater than $\frac{1}{n}$. A nice trick is to let

$$\epsilon_i = p_i - \frac{1}{n} \text{ for } i = 1, ..., n.$$

Then

$$p_1^2 + \cdots + p_n^2 = (\epsilon_1 + \frac{1}{n})^2 + \cdots + (\epsilon_n + \frac{1}{n})^2 = \epsilon_1^2 + \cdots + \epsilon_n^2 + \frac{2}{n}(\epsilon_1 + \cdots + \epsilon_n) + \frac{1}{n}.$$

Now, since $p_1 + \cdots + p_n = 1$, we can conclude that $\epsilon_1 + \cdots + \epsilon_n = 0$. Hence,

$$p_1^2 + \cdots + p_n^2 = \epsilon_1^2 + \cdots + \epsilon_n^2 + \frac{1}{n} > \frac{1}{n}$$

precisely when not all the ϵ_i's are zero, i.e. when the die is "fixed".

40. *Is it possible to have a non-fair six-sided die such that the probability of rolling* $2, 3, 4, 5,$ *and* 6 *is the same whether we roll it once or twice (and sum)? What about for other numbers of sides?*

Let's start with a 2-sided die.

Suppose the probability of rolling a one is a_1 and the probability of rolling a 2 is a_2. Then the probability of rolling a 2 when rolling twice and summing is a_1^2.

So, to achieve equal probabilities whether rolling once or twice, we require non-negative a_1 and a_2 with

$$a_1 + a_2 = 1 \text{ and } a_1^2 = a_2$$

so that $a_1^2 + a_1 - 1 = 0$, and hence

$$a_1 = \frac{-1+\sqrt{5}}{2} = \frac{1}{\phi} \approx 0.6180339887498, \text{ and } a_2 = \frac{3-\sqrt{5}}{2} = \frac{1}{\phi^2} \approx 0.381966011250.$$

where ϕ is the golden ratio.

Let's look at the six-sided case. Here, we seek $a_1, \ldots, a_6$ non-negative with $a_1 + \cdots + a_6 = 1$ and

$$\begin{aligned} a_1^2 &= a_2 \\ 2a_1a_2 &= a_3 \\ 2a_1a_3 + a_2^2 &= a_4 \\ 2a_1a_4 + 2a_2a_3 &= a_5 \\ 2a_1a_5 + 2a_2a_4 + a_3^2 &= a_6. \end{aligned}$$

This implies

$$\begin{aligned} a_1^2 &= a_2 \\ 2a_1^3 &= a_3 \\ 5a_1^4 &= a_4 \\ 14a_1^5 &= a_5 \\ 42a_1^6 &= a_6 \end{aligned}$$

and so we seek a (real, bounded between zero and 1) solution to the polynomial

$$42a_1^6 + 14a_1^5 + 5a_1^4 + 2a_1^3 + a_1^2 + a_1 - 1.$$

This polynomial has one positive real root: $a_1 = 0.3833276422504671918282678397....$ Thus, weighting a die with probabilities

$$\begin{aligned} a_1 &= 0.3833276422504671918282678397\ldots \\ a_2 &= 0.1469400813133021601465169741\ldots \\ a_3 &= 0.1126523898438400996320617058\ldots \\ a_4 &= 0.1079569374817992533848329781\ldots \\ a_5 &= 0.1158720592665417507230810930\ldots \\ a_6 &= 0.1332508898440495442852394095\ldots \end{aligned}$$

will give us a die such that the probability of rolling $2, 3, 4, 5$ and 6 is the same whether we roll once or roll twice and sum.

In general, for an n-sided die, we require $a_1, \ldots, a_n$ non-negative real numbers with

$$\sum_{i=1}^{n} a_i = 1$$

and

$$C_{i-1} a_1^i = a_i, i = 1, \ldots, n$$

where C_i is the i-th Catalan number, $C_i = \frac{\binom{2i}{i}}{i+1}$. Thus we need to guarantee a positive real root less the one for the polynomial

$$-1 + \sum_{i=1}^{n} C_{i-1} x^i$$

Since this polynomial evaluates to -1 at $x = 0$, and is positive for $x = 1$, there must be at least one real solution between 0 and 1, and so there exists one of these special dice for every number of sides.

For an n-sided die, we have the following approximate a_1 values:

n	a_1
2	0.6180339887498
3	0.5
4	0.4418115119484
5	0.4068294316935
6	0.3833276422504
7	0.3663733452433
8	0.3535209284167
9	0.3434158345289
10	0.3352452267388
20	0.2969330618649
50	0.2714018346938
100	0.2617716572724
200	0.2564370408369
500	0.2528692107822
1000	0.2515454964644

Some observations (proofs to be added later):

(a) It appears that a_1 approaches $\frac{1}{4}$ as n tends to infinity.

(b) For any given number of sides n, it appears that the values of a_i decreases as i increases until reaching a minimum with $i = \lceil \frac{n}{2} \rceil$, and increases thereafter.

41. *Find a pair of 6-sided dice, labelled with positive integers differently from the standard dice, so that the sum probabilities are the same as for a pair of standard dice.*

Number one die with sides 1,2,2,3,3,4 and one with 1,3,4,5,6,8. Rolling these two dice gives the same sum probabilities as two normal six-sided dice.

A natural question is: how can we find such dice? One way is to consider the polynomial

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2.$$

This factors as

$$x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2.$$

We can group this factorization as

$$\left(x(1+x)(1+x+x^2)\right)\left(x(1+x)(1+x+x^2)(1-x+x^2)^2\right)$$
$$=(x+2x^2+2x^3+x^4)(x+x^3+x^4+x^5+x^6+x^8).$$

This yields the "weird" dice (1,2,2,3,3,4) and (1,3,4,5,6,8). See Appendix C for more about this method.

See [1] for more on renumbering dice.

42. *Is it possible to have two non-fair n-sided dice, with sides numbered 1 through n, with the property that their sum probabilities are the same as for two fair n-sided dice?*

Another way of asking the question is: suppose you are given two n-sided dice that exhibit the property that when rolled, the resulting sum, as a random variable, has the same probability distribution as for two fair n-sided dice; can you then conclude that the two given dice are fair? This question was asked by Lewis Robertson, Rae Michael Shortt and Stephen Landry in [4]. Their answer is surprising: you *can* sometimes, depending on the value of n. Specifically, if n is 1,2,3,4,5,6,7,8,9,11 or 13, then two n-sided dice whose sum "acts fair" are, in fact, fair. If n is any other value, then there exist pairs of n-sided dice which are not fair, yet have "fair" sums.

The smallest example, with $n = 10$, gives dice with the approximate probabilities (see [Rob 2] for the exact values)

$$(0.07236, 0.14472, 0.1, 0.055279, 0.127639, 0.127639, 0.055279, 0.1, 0.14472, 0.07236)$$

and

$$(0.13847, 0, 0.2241, 0, 0.13847, 0.13847, 0, 0.2241, 0, 0.13847).$$

It's clear that these dice are not fair, yet the sum probabilities for them are the same as for two fair 10-sided dice.

43. *Is it possible to have two non-fair 6-sided dice, with sides numbered 1 through 6, with a uniform sum probability? What about n-sided dice?*

No. Let p_1, p_2, p_3, p_4, p_5 and p_6 be the probabilities for one 6-sided die, and q_1, q_2, q_3, q_4, q_5 and q_6 be the probabilities for another. Suppose that these dice together yield sums with uniform probabilities. That is, suppose $P(\text{sum} = \text{k}) = \frac{1}{11}$ for $k = 2, ..., 12$. Then

$$p_1 q_1 = \frac{1}{11} \text{ and } \mathrm{p_6 q_6} = \frac{1}{11}.$$

Also,

$$\frac{1}{11} = P(\text{sum} = 7) \geq \mathrm{p_1 q_6 + p_6 q_1}$$

so

$$p_1 \frac{1}{11 p_6} + p_6 \frac{1}{11 p_1} \leq \frac{1}{11}$$

i.e.,

$$\frac{p_1}{p_6} + \frac{p_6}{p_1} \leq 1.$$

Now, if we let $x = \frac{p_1}{p_6}$, then we have

$$x + \frac{1}{x} \leq 1$$

which is impossible, since for positive real x, $x + \frac{1}{x} \geq 2$. Thus, no such dice are possible.

An identical proof shows that this is an impossibility regardless of the number of sides of the dice.

44. *Suppose that we renumber three fair 6-sided dice (A, B, C) as follows: $A = \{2, 2, 4, 4, 9, 9\}$, $B = \{1, 1, 6, 6, 8, 8\}$, and $C = \{3, 3, 5, 5, 7, 7\}$.*

    (a) *Find the probability that die A beats die B; die B beats die C; die C beats die A.*

    (b) *Discuss.*

    The probability that A beats B can be expressed as

    $$\left(\frac{2}{6}\right)\left(\frac{2}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{2}{6}\right) + \left(\frac{2}{6}\right)(1) = \frac{5}{9}.$$

    The thinking behind this goes like this: the probability of rolling a 2 with A is 2/6, and if a 2 is rolled, it will beat B with probability 2/6. The probability of rolling a 4 with A is 2/6, and it will beat B with probability 2/6. The probability of rolling a 9 with A is 2/6, if it will beat B with probability 1.

    Similarly, the probability that B beats C is

    $$\left(\frac{2}{6}\right)\left(\frac{4}{6}\right) + \left(\frac{2}{6}\right)(1) = \frac{5}{9}$$

    and the probability that C beats A is

    $$\left(\frac{2}{6}\right)\left(\frac{2}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{4}{6}\right) + \left(\frac{2}{6}\right)\left(\frac{4}{6}\right) = \frac{5}{9}.$$

    Thus, each die beats another with probability greater than 50%. This is certainly a counterintuitive notion; this shows that "beats", as in "die 1 beats die 2" is not transitive.

    Lots of questions arise. What other sets of "non-transitive" dice are possible? What is the fewest number of sides necessary? For a given number of sides, what is the minimum possible maximum face value (e.g., in the set given above, the maximum face value is 9)? For a given number of sides, and a bound on the face values, how many sets of transitive dice are there? What about sets with more than three dice?

45. *Find every six-sided die with sides numbered from the set $\{$1,2,3,4,5,6$\}$ such that rolling the die twice and summing the values yields all values between 2 and 12 (inclusive). For instance, the die numbered 1,2,4,5,6,6 is one such die. Consider the sum probabilities of these dice. Do any of them give sum probabilities that are "more uniform" than the sum probabilities for a standard die? What if we renumber two dice differently - can we get a uniform (or more uniform than standard) sum probability?*

    The numbers 1, 2, 5 and 6 must always be among the numbers on the die, else sums of 2, 3, 11 and 12 would not be possible. In order to get a sum of 5, either 3 or 4 must be on the die also. The last place on the die can be any value in $\{$1,2,3,4,5,6$\}$. Hence there are 11 dice with the required property. Listed with their corresponding error, they are:

1,2,4,5,6,6	0.0232884399551066
1,2,4,5,5,6	0.0325476992143659
1,2,4,4,5,6	0.0294612794612795
1,2,3,5,6,6	0.0232884399551066
1,2,3,5,5,6	0.026374859708193
1,2,3,4,5,6	0.0217452300785634
1,2,3,3,5,6	0.0294612794612795
1,2,2,4,5,6	0.026374859708193
1,2,2,3,5,6	0.0325476992143659
1,1,2,4,5,6	0.0232884399551066
1,1,2,3,5,6	0.0232884399551066

The error here is the sum of the square of the difference between the actual probability of rolling each of the sums 2 through 12 and 1/11 (the probability we would have for each sum if we had a uniform distribution). That is, if p_i is the probability of rolling a sum of i with this die, then the error is

$$\sum_{i=2}^{12} \left(p_i - \frac{1}{11} \right)^2 .$$

Note that the standard die gives the smallest error (i.e., the closest to uniform sum).

If we renumber two dice differently, many more cases are possible. One pair of dice are 1,3,4,5,6,6 and 1,2,2,5,6,6. These two dice give all sum values between 2 and 12, with an error (as above) of 0.018658810325477, more uniform than the standard dice. The best dice for near-uniformity are 1,2,3,4,5,6 and 1,1,1,6,6,6 which yield all the sums from 2 to 12 with near equal probability: the probability of rolling 7 is 1/6 and all other sums are 1/12. The error is 5/792, or about 0.00631.

3.4 Games with Dice

46. *Two players each roll a die. Player 1 rolls a fair m-sided die, while player 2 rolls a fair n sided die, with $m > n$. The winner is the one with the higher roll. What is the probability that Player 1 wins? What is the probability that Player 2 wins? What is the probability of a tie? If the players continue rolling in the case of a tie until they do not tie, which player has the higher probability of winning? If the tie means a win for Player 1 (or player 2), what is their probability of winning?*

When the two players roll there dice, there are mn possible outcomes. These can be thought of as lattice points, i.e., points (x, y) in the xy-plane where x and y are positive integers, and $1 \leq x \leq m$ and $1 \leq y \leq n$.

Of these mn lattice points, Player 1 is a winner whenever $x > y$. The number of lattice points with $x > y$ is equal to

$$(m-1) + (m-2) + (m-3) + \cdot + (m-n) = nm - \frac{1}{2}n(n+1).$$

Hence, the probability of Player 1 winning is

$$1 - \frac{n+1}{2m}.$$

Of the nm possibly outcomes, n are ties. So the probability of a tie is

$$\frac{n}{mn} = \frac{1}{m}.$$

Hence, the probability of Player 2 winning is

$$\frac{n-1}{2m}.$$

Now, suppose the players continue rolling until there is no tie, and then the winner is declared based on the final roll. What are the winning probabilities then? Let p be the probability of Player 1 rolling a larger roll on a single roll, and q be the probability of a tie. Then, for Player 1 to win, there must be a sequence of ties, followed by a single winning roll by Player 1. Hence, the probability of Player 1 winning is

$$\sum_{i=0}^{\infty} q^i p = p \sum_{i=0}^{\infty} q^i = \frac{p}{1-q}.$$

Using the values for p and q calculated above, we find that the probability of Player 1 winning if the players reroll until they are not tied is

$$\frac{1-\frac{n+1}{2m}}{1-\frac{1}{m}} = 1 - \frac{n-1}{2(m-1)}$$

and so the probability of Player 2 winning is

$$\frac{n-1}{2(m-1)}.$$

If, instead of rerolling, ties mean a win for Player 1, then Player 1's probability of winning becomes

$$1 - \frac{n+1}{2m} + \frac{1}{m} = 1 - \frac{n-1}{2m}.$$

while Player 2's is

$$\frac{n-1}{2m}.$$

If, instead of rerolling, ties mean win for Player 2, then Player 1's probability of winning is

$$1 - \frac{n+1}{2m}.$$

while Player 2's is

$$\frac{n-1}{2m} + \frac{1}{m} = \frac{n+1}{2m}.$$

Since $\frac{n+1}{2m} < \frac{1}{2}$ as long as $n < m-1$, Player 2 is at a disadvantage even if ties go them, except when $n = m-1$, in which case the players are evenly matched (each with a $\frac{1}{2}$ probability of winning). If Player 1's die has at least two more faces than Player 2's die, Player 1 has the advantage, regardless of how ties are treated.

47. **Craps** *What is the probability of winning a round of the game Craps?*

The probability of winning a round of craps can be expressed as

$$P(\text{rolling 7 or 11}) + \sum_{b \in \{4,5,6,8,9,10\}} P(\text{rolling } b) P(\text{rolling } b \text{ again before rolling } 7).$$

We now evaluate each probability. The probability of rolling 7 is $\frac{6}{36} = \frac{1}{6}$, and the probability of rolling 11 is $\frac{2}{36} = \frac{1}{18}$. Hence,

$$P(\text{rolling } 7 \text{ or } 11) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}.$$

The following table gives the probability of rolling b, for $b \in \{4, 5, 6, 8, 9, 10\}$. (This is the probability of b becoming the "point".)

4	3/36 = 1/12
5	4/36 = 1/9
6	5/36
8	5/36
9	4/36 =1/9
10	3/36 =1/12

Finally, we need to determine the probability of rolling b before rolling 7. Let p be the probability of rolling b on any single roll. Rolling b before rolling 7 involves rolling some number of rolls, perhaps zero, which are not b or 7, followed by a roll of b. The probability of rolling k rolls which are not b or 7, followed by a roll of b is

$$\left(1 - p - \frac{1}{6}\right)^k p = \left(\frac{5}{6} - p\right)^k p.$$

Since k may be any non-negative integer value, we have

$$P(\text{rolling } b \text{ before rolling } 7) = \sum_{k=0}^{\infty} \left(\frac{5}{6} - p\right)^k p = \frac{p}{\frac{1}{6} + p}.$$

See Appendix B for some formulas for simplifying series such as the one above. Another way of looking at this is that the probability of rolling b before rolling a 7 is the conditional probability of rolling b, given that either b or 7 was rolled.

We can calculate the following table:

b	$P(\text{rolling } b)$	$P(\text{rolling } b \text{ again before rolling } 7)$	$P(\text{rolling } b)P(\text{rolling } b \text{ again before rolling } 7)$
4	1/12	1/3	1/36
5	1/9	2/5	2/45
6	5/36	5/11	25/396
8	5/36	5/11	25/396
9	1/9	2/5	2/45
10	1/12	1/3	1/36

Thus, the probability of winning a round of craps is

$$\frac{2}{9} + \frac{1}{36} + \frac{2}{45} + \frac{25}{396} + \frac{25}{396} + \frac{2}{45} + \frac{1}{36} = \frac{244}{495} = 0.49\overline{29}.$$

Since $\dfrac{244}{495} = \dfrac{1}{2} - \dfrac{7}{990}$, the odds are just slightly against the player.

48. **Non-Standard Craps** *We can generalize the games of craps to allow dice with other than six sides. Suppose we use two (fair) n-sided dice. Then we can define a game analogous to craps in the following way. The player rolls two n-sided dice. If the sum of these dice is $n+1$ or $2n-1$, the player wins. If the sum of these dice is $2, 3$ or $2n$ the player loses. Otherwise the sum becomes the*

player's point, and they win if they roll that sum again before rolling $n+1$. We may again ask: what is the player's probability of winning?

For two n-sided dice, the probability of rolling a sum of $n+1$ is

$$P(n+1) = \frac{n}{n^2} = \frac{1}{n}$$

and the probability of rolling a sum of $2n-1$ is

$$P(2n-1) = \frac{2}{n^2}.$$

In general, the probability of a sum of k is

$$P(k) = \frac{n - |k-n-1|}{n^2}.$$

Hence, the probability of winning a round of Craps with n-sided dice is

$$p_n = \frac{1}{n} + \frac{2}{n^2} + \sum_{\substack{4 \le k \le 2n-2 \\ k \ne n+1}} \frac{P(k)^2}{P(k)+P(n+1)} = \frac{1}{n} + \frac{2}{n^2} + \sum_{\substack{4 \le k \le 2n-2 \\ k \ne n+1}} \frac{(n-|k-n-1|)^2}{n^2(2n-|k-n-1|)}$$

We have the following table:

n	p_n
3	5/9 = 0.55555...
4	15/28 = 0.535714...
5	461/900 =0.512222...
6	244/495 = 0.492929...
7	100447/210210 = 0.477841...
8	37319/80080 = 0.4660214...
9	2288779/5012280 = 0.456634...
10	23758489/52907400 = 0.449057...
20	0.415459...
30	0.404973...
50	0.397067...
100	0.391497...
1000	0.386796...
10000	0.386344...
100000	0.386299...
1000000	0.38629486...

It certainly appears that p_n approaches a limit as n approaches infinity.

49. **Yahtzee** *There are many probability questions we may ask with regard to the game of Yahtzee. For starters, what is the probability of rolling, in a single roll,*

    a) Yahtzee

    b) Four of a kind (but not Yahtzee)

    c) A full house

    d) Three of a kind (but not Yahtzee, four of a kind or full house)

    e) A long straight

    f) A small straight

    These questions aren't *too* tricky, so I'll just give the probabilities here:

(a) Yahtzee: $\frac{6}{6^5} = \frac{1}{1296} \approx 0.07716\%$

(b) Four of a kind (but not Yahtzee): $\frac{\binom{5}{4} \cdot 6 \cdot 5}{6^5} = \frac{25}{1296} \approx 1.929\%$

(c) A full house: $\frac{\binom{5}{3} \cdot 6 \cdot 5}{6^5} = \frac{25}{648} \approx 3.858\%$

(d) Three of a kind (but not Yahtzee, four of a kind or full house) :

$$\frac{\binom{5}{3} \cdot 6 \cdot 5 \cdot 4}{6^5} = \frac{25}{162} \approx 15.432\%$$

(e) A long straight: $\frac{2 \cdot 5!}{6^5} = \frac{5}{162} \approx 3.086\%$

(f) A small straight (but not a long straight):

$$\frac{\frac{5!}{2!} \cdot 4 + 2\left(\frac{5!}{2!} \cdot 4 + 5!\right)}{6^5} = \frac{10}{81} \approx 12.346\%$$

50. **More Yahtzee** *What is the probability of getting Yahtzee, assuming that we are trying just to get Yahtzee, we make reasonable choices about which dice to re-roll, and we have three rolls? That is, if we're in the situation where all we have left to get in a game of Yahtzee is Yahtzee, so all other outcomes are irrelevant.*

This is quite a bit trickier than the previous questions on Yahtzee. The difficulty here lies in the large number of ways that one can reach Yahtzee: roll it on the first roll; roll four of a kind on the first roll and then roll the correct face on the remaining die, etc. One way to calculate the probability is to treat the game as a *Markov chain* (see Appendix D for general information on Markov chains).

We consider ourselves in one of five *states* after each of the three rolls. We will say that we are in state b if we have b common dice among the five. For example, if a roll yields 12456, we'll be in state 1; if a roll yields 11125, we'll be in state 3. Now, the goal in Yahtzee is to try to get to state 5 in three rolls (or fewer). Each roll gives us a chance to change from our initial state to a better, or equal, state. We can determine the probabilities of changing from state i to state j. Denote this probability by $P_{i,j}$. Let the 0 state refer to the initial state before rolling. Then we have the following probability matrix:

$$P = (P_{i,j}) = \begin{pmatrix} 0 & \frac{120}{1296} & \frac{900}{1296} & \frac{250}{1296} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & \frac{120}{1296} & \frac{900}{1296} & \frac{250}{1296} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & 0 & \frac{120}{216} & \frac{80}{216} & \frac{15}{216} & \frac{1}{216} \\ 0 & 0 & 0 & \frac{25}{36} & \frac{10}{36} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.16}$$

The one representing $P_{5,5}$ indicates that if we reach yahtzee, state 5, before the third roll, we simply stay in that state. Now, the probability of being in state 5 after 3 rolls is given by

$$\sum P_{0,i_1} P_{i_1,i_2} P_{i_2,5} = (M^3)_{1,5}$$

where the sum is over all triples (i_1, i_2, i_3) with $0 \le i_j \le 5$. Calculating M^3 gives us the probability

$$\frac{2783176}{6^{10}} = \frac{347897}{7558272} \approx 0.04603.$$

Since $\frac{347897}{7558272} = \frac{1}{21.7256026...}$, a player will get Yahtzee about once out of every twenty two attempts.

51. **Drop Dead**

(a) *What is the expected value of a player's score?*

(b) *What is the probability of getting a score of* 0*?* 1*?* 10*?* 20*? etc.*

(a) The player begins with five dice, and throws them repeatedly, until no dice are left. The key factor in calculating the expected score is the fact that the number of dice being thrown changes. When throwing n dice, a certain number may "die" (i.e. come up 2 or 5), and leave j non-dead dice. The probability of this occurring is

$$P_{n,j} = \binom{n}{n-j} \frac{2^{n-j} 4^j}{6^n}.$$

The following table gives $P_{n,j}$ for n and j between 0 and 5.

n\j	0	1	2	3	4	5
1	1/3	2/3	0	0	0	0
2	1/9	4/9	4/9	0	0	0
3	1/27	6/27	12/27	8/27	0	0
4	1/81	8/81	24/81	32/81	16/81	0
5	1/243	10/243	40/243	80/243	80/243	32/243

When throwing n dice, the expected sum is $3.5n$, if none of the dice come up 2 or 5. Let $E(n)$ represent the expected score starting with n dice (so we're ultimately concerned with $E(5)$). Consider $E(1)$. Rolling a single die, the expected score is

$$E(1) = 3.5P_{1,1} + E(1)P_{1,1}.$$

That is, in one roll, we pick up 3.5 points, on average, if we don't "drop dead" (so we get $3.5P_{1,1}$ expected points), and then we're in the same position as when we started (so we pick up $E(1)P_{1,1}$ expected points). We can solve this equation to get

$$E(1) = 3 \cdot \left(\frac{2}{3}\right)\left(\frac{7}{2}\right) = 7.$$

Now, suppose we start with 2 dice. The expected score is

$$E(2) = (2 \cdot 3.5 + E(2))\, P_{2,2} + E(1)P_{2,1}.$$

That is, on a single roll, we pick up $2 \cdot 3.5$ points on average if none of the dice "die", in which case we're back where we started from (and then expect to pick up E(2) points), *or* exactly one of the dice "die", and so we expect to pick up $E(1)$ points with the remaining die. This equation yields

$$E(2) = \frac{1}{1 - P_{2,2}} \left(7P_{2,2} + E(1)P_{2,1}\right) = \frac{56}{5}.$$

Continuing in this way, we have the general equation

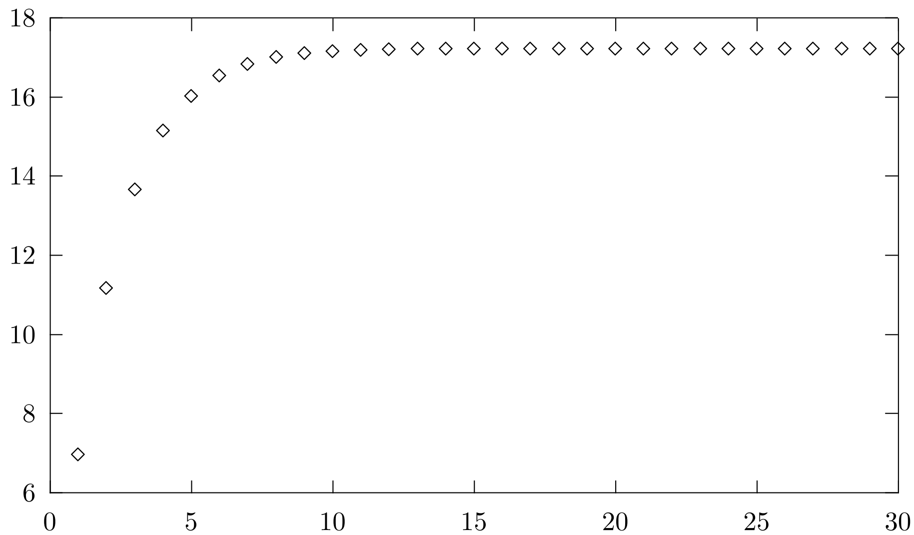
$$E(n) = 3.5 \cdot n \cdot P_{n,n} + \sum_{j=1}^{n} E(j)P_{n,j}$$

which we can rewrite as

$$E(n) = \frac{1}{1 - P_{n,n}} \left(3.5 \cdot n \cdot P_{n,n} + \sum_{j=1}^{n-1} E(j)P_{n,j}\right)$$

With this formula, we can calculate $E(n)$ for whatever value of n we want. Here is a table of $E(n)$:

n	$E(n)$
1	7
2	$\frac{56}{5} = 11.2$
3	$\frac{1302}{95} \approx 13.70526$
4	$\frac{3752}{247} \approx 15.19028$
5	$\frac{837242}{52117} \approx 16.06466$
6	$\frac{4319412}{260585} \approx 16.57583$
10	$\frac{9932935945754444}{577645434482545} \approx 17.19556$
20	≈ 17.26399
30	$\approx 17.264123714008007018098412\,13$
100	$\approx 17.26412423601867057324993502$
250	$\approx 17.26412422187783220247082379$



So we see that a game of Drop Dead, using 5 dice, will have, on average, a score of about 16.06.

Further questions: Notice that if we play the game with more than 5 dice, the expected score does not increase very much. In fact, it appears as if there is an *upper bound* on the expected score; that is, it seems that there is some B so that $E(n) < B$ for all n. What is the smallest possible value for B? Also, we expect $E(n)$ to always increase as n increases. Can we prove this is so?

(b) Calculating the exact probabilities of scores seems to be a bit of a pain. The easiest score to work out is zero. To get zero, the player must roll at least one 2 or 5 on every roll. If we define a Markov process, with states 0, F, 5, 4, 3, 2, 1 (in that order), where 0 means a score of zero has been achieved, F means a score greater than 0 has been achieved, and 5 through 1 are the current number of dice being rolled, we have the following transition matrix:

$$P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{243} & \frac{32}{243} & 0 & \frac{80}{243} & \frac{80}{243} & \frac{40}{243} & \frac{10}{243} \\ \frac{1}{81} & \frac{16}{81} & 0 & 0 & \frac{32}{81} & \frac{8}{27} & \frac{8}{81} \\ \frac{1}{27} & \frac{8}{27} & 0 & 0 & 0 & \frac{4}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{4}{9} & 0 & 0 & 0 & 0 & \frac{4}{9} \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the games takes at most five rolls, the fifth power of this matrix tells us what we want to know:

$$P_0^5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{978853}{4782969} & \frac{3804116}{4782969} & 0 & 0 & 0 & 0 & 0 \\ \frac{4163}{19683} & \frac{15520}{19683} & 0 & 0 & 0 & 0 & 0 \\ \frac{55}{243} & \frac{188}{243} & 0 & 0 & 0 & 0 & 0 \\ \frac{7}{27} & \frac{20}{27} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus we see that the probability of achieving a score of zero is $\frac{978853}{4782969} = \frac{978853}{3^{14}}$, which is about $0.2046538457598\ldots$.

The probability of achieving a score of 1 is calculable in a similar way. Our transition matrix is

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{11}{81} & 0 & \frac{80}{243} & \frac{80}{243} & \frac{40}{243} & \frac{10}{243} \\ 0 & \frac{17}{81} & 0 & 0 & \frac{32}{81} & \frac{8}{27} & \frac{8}{81} \\ 0 & \frac{1}{3} & 0 & 0 & 0 & \frac{4}{9} & \frac{2}{9} \\ 0 & \frac{5}{9} & 0 & 0 & 0 & 0 & \frac{4}{9} \\ \frac{1}{18} & \frac{17}{18} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

A note on this lower-left most entry: once the player has only one die left, they have a 1/6 chance of rolling a one; but then, the die must die, which occurs with probability 1/3. Hence the 1/18 probability of getting a score of 1 after the state of one die is attained.

Raising this matrix to the fifth power yields

$$P_1^5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{305285}{14348907} & \frac{4477684}{4782969} & 0 & 0 & 0 & 0 & 0 \\ \frac{1300}{59049} & \frac{18383}{19683} & 0 & 0 & 0 & 0 & 0 \\ \frac{17}{729} & \frac{226}{243} & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{81} & \frac{25}{27} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{18} & \frac{17}{18} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and thus the probability of a score of 1 in this game is $305285/14348907 = 0.0212758365497804118\ldots$.

Scores higher than 1 are more difficult, since it will not be necessary to reach a single die. On the other hand, to achieve a score of n, there can be at most $n+4$ rolls, so the problem is finite. Appendix F has experimental data for comparison to any exact calculations you might attempt.

52. **Threes** *In the game of Threes, the player starts by rolling five standard dice. In the game, the threes count as zero, while the other faces count normally. The goal is to get as low a sum as possible. On each roll, at least one die must be kept, and any dice that are kept are added to the player's sum. The game lasts at most five rolls, and the score can be anywhere from 0 to 30.*

For example a game might go like this. On the first roll the player rolls

$$2-3-3-4-6$$

The player decides to keep the 3s, and so has a score of zero. The other three dice are rolled, and the result is

$$1-5-5$$

Here the player keeps the 1, so their score is 1, and re-rolls the other two dice. The result is

$$1-2$$

Here, the player decides to keep both dice, and their final score is 4.

If a player plays optimally (i.e., using a strategy which minimizes the expected value of their score), what is the expected value of their score?

This is certainly best analysed in reverse.

Suppose we are rolling one die. Then the expected value of the result is

$$\frac{1+2+0+4+5+6}{6}=3.$$

Suppose we roll two dice. The rules require that we keep at least one, so clearly we must keep the lower of the two. The question is whether to keep the other one. If we don't keep it, our expected value from it will be 3 when we reroll. Hence, we should keep it if it is a 3, a 1, or a 2.

Following this method, the expected value with two dice is expressible as

$$E_2=\frac{1}{36}\sum_{i=1}^{6}\sum_{j=1}^{6}\left(\min\{i,j\}+\min\{\max\{i,j\},3\}\right)=\frac{158}{36}=\frac{79}{18}=4.388\ldots.$$

Suppose we roll three dice. We must keep the lowest die, so we need to decide whether to keep either of the other two dice. Obviously, if we keep only one of them, we would keep the lower one. Call the three dice

$$d_1\leq d_2\leq d_3.$$

Then if we keep both d_2 and d_3, our sum is d_2+d_3. If we re-roll only d_3, then our expected sum is d_2+3. If we re-roll both d_2 and d_3, then our expected sum is 4.3888... . Thus we want to choose the option so that our expected sum is

$$\min\{d_2+d_3,d_2+3,E_2\}.$$

Analyzing this, we find that if $d_2\geq 4$, we should re-roll both. If $d_2=3$, we should keep both if $d_3<3$. If $d_2=2$, then we should keep d_3 if $d_3=2$; otherwise we should re-roll both. (This is the surprising part of the optimal strategy: a two is not necessarily keepable by itself, but depends on the value of other die.) If $d_2=1$ and $d_3=1$ or 2, keep both; otherwise, keep d_2 and re-roll d_3.

The calculation of the expected value with three dice can be expressed as

$$E_3=\frac{1}{6^3}\sum_{\substack{i,j,k\\=0}}^{6}{}'\left(d_1+\min\{d_2+d_3,d_2+3,E_2\}\right)=\frac{2261}{2\cdot 6^3}=5.233796...$$

where the sum skips 3, and $d_1\leq d_2\leq d_3$ is $\{i,j,k\}$ sorted in increasing order.

Continuing in this way, the expected value with four dice can be expressed as

$$E_4=\frac{1}{6^4}\sum_{\substack{i,j,k,l\\=0}}^{6}{}'\left(d_1+\min\{d_2+d_3+d_4,d_2+d_3+3,d_2+E_2,E_3\}\right)=\frac{1663107}{6^7}=5.833858...$$

where the sum skips 3, and $d_1 \leq d_2 \leq d_3 \leq d_4$ is $\{i, j, k, l\}$ sorted in increasing order.

Finally, the expected value with five dice can be expressed as

$$\begin{aligned} E_5 &= \frac{1}{6^5} \sum_{\substack{i,j,k,l,m \\ =0}}^{6}{}' \left(d_1 + \min \left\{ \sum_{n=2}^{5} d_n, d_2 + d_3 + d_4 + 3, d_2 + d_3 + E_2, d_2 + E_3, E_4 \right\} \right) \\ &= \frac{13613549985}{6^{12}} = 6.253978525.... \end{aligned}$$

where the sum skips 3, and $d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_5$ is $\{i, j, k, l, m\}$ sorted in increasing order.

Thus, the expected score in this game, played with an optimal strategy, is about 6.25398.

But, what is the optimal strategy? It is essentially encoded above, but is there a more simple statement? A small simplification is made by noting that a sum of integers is less than, say, 4.38888... only if the sum is less than or equal to 4. So the E_i values that appear in the sums above can be replaced by their integer parts.

It is a tricky strategy to paraphrase. Consider that if you roll 3-3-3-2-2, you should keep all the dice, but if you roll 3-3-2-2-2, you should re-roll the 2s, since $6 > E_3 = 5.233796...$. The strategy is not summarizable to a “this die or less should always be kept on roll i” simplicity.

A further question: what is the probability of getting a score of zero? This question has more than one interpretation: (a) what is the probability of getting a score of zero if played using the “optimal” strategy above, and (b) what is the probability of getting a score of zero if the player does everything possible to get a score of zero (i.e., keeps only 3s as long as possible).

(Special thanks to David Korsnack for inspiring me to look into this problem, and for providing some numerics with which I could compare my calculations.)

53. *Suppose we play a game with a die where we roll and sum our rolls as long as we keep rolling larger values. For instance, we might roll a sequence like 1-3-4 and then roll a 2, so our sum would be 8. If we roll a 6 first, then we're through and our sum is 6. Three questions about this game:*

    (a) What is the expected value of the sum?

    (b) What is the expected value of the number of rolls?

    (c) If the game is played with an n-sided die, what happens to the expected number of rolls as n approaches infinity?

We can consider this game as a Markov chain with an absorbing state. If we consider the state to be the value of the latest roll, or 7 if the latest roll is not larger than the previous one, then we have the following transition matrix:

$$P = \begin{pmatrix} 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 2/6 \\ 0 & 0 & 0 & 1/6 & 1/6 & 1/6 & 3/6 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 & 4/6 \\ 0 & 0 & 0 & 0 & 0 & 1/6 & 5/6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.17}$$

Using the notation of Appendix D, we have

$$Q = \begin{pmatrix} 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so that $N = (I - Q)^{-1}$ is

$$N = \begin{pmatrix} 1 & 1/6 & 7/36 & 49/216 & 343/1296 & 2401/7776 \\ 0 & 1 & 1/6 & 7/36 & 49/216 & 343/1296 \\ 0 & 0 & 1 & 1/6 & 7/36 & 49/216 \\ 0 & 0 & 0 & 1 & 1/6 & 7/36 \\ 0 & 0 & 0 & 0 & 1 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The row sum of N is

$$\begin{pmatrix} 16807/7776 \\ 2401/1296 \\ 343/216 \\ 49/36 \\ 7/6 \\ 1 \end{pmatrix}$$

and so the expected number of rolls before absorption (i.e., the number of rolls that count in the sum) is

$$(1/6)\,(16807/7776 + 2401/1296 + 343/216 + 49/36 + 7/6 + 1) = 70993/7776 \approx 1.521626.$$

We use N to calculate the expected sum as well. If the first roll is a 1, the expected sum will be

$$1 + 2\cdot\frac{1}{6} + 3\cdot\frac{7}{36} + 4\cdot\frac{49}{216} + 5\cdot\frac{343}{1296} + 6\cdot\frac{2401}{7776} = 6.$$

In fact, for any first roll, the expected sum is 6. Hence, the expected sum is 6.

Now, suppose the game is played with an n-sided die. Let E be the expected number of rolls. Let $E(j)$ be the expected number of rolls if the first roll is j. Then,

$$E(j) = 1 + \frac{1}{n}E(j+1) + \frac{1}{n}E(j+2) + \cdots + \frac{1}{n}E(n)$$

and so

$$E(j+1) = 1 + \frac{1}{n}E(j+2) + \frac{1}{n}E(j+3) + \cdots + \frac{1}{n}E(n)$$

from which we can conclude

$$E(j) = \left(1 + \frac{1}{n}\right)E(j+1).$$

Since $E(n) = 1$, we have

$$E(j) = \left(1 + \frac{1}{n}\right)^{n-j}.$$

Thus,

$$E = \frac{1}{n}\sum_{j=1}^{n} E(j) = \frac{1}{n}\sum_{j=1}^{n}\left(1 + \frac{1}{n}\right)^{n-j} = \frac{1}{n}\left(\frac{n+1}{n}\right)^{n}\sum_{j=1}^{n}\left(\frac{n}{n+1}\right)^{j}$$

$$= \left(\frac{n+1}{n}\right)^n \left(1 - \left(\frac{n}{n+1}\right)^n\right) = \left(\frac{n+1}{n}\right)^n - 1.$$

And so we see that

$$\lim_{n\to\infty} E = e - 1 = 1.718281828459....$$

54. *Suppose we play a game with a die where we roll and sum our rolls. We can stop any time, and the sum is our score. However, if our sum is ever a multiple of* 10*, our score is zero, and our game is over. What strategy will yield the greatest expected score? What about the same game played with values other than* 10*?*

Let's generalize things right away.

Suppose we want to avoid multiples of m. For now, let's suppose $m \geq 6$.

Suppose we have been playing, and our sum is n, and

$$km < n < (k+1)m$$

for some integer $k > 0$.

If our score is less than $(k+1)m - 6$, then we should certainly roll until it is at least $(k+1)m - 6$, since there is no risk.

Suppose $(k+1)m - 6 \leq n < (k+1)m$. Should we roll to get past $(k+1)m$? There is at least a $1/6$ chance that this will result in a score of zero, and the best our score could be (without risking another multiple of m) is $(k+2)m - 1$. Hence the expected value E of our score with risking one multiple of m is

$$E < \frac{5}{6}((k+2)m - 1)$$

and this is less than $(k+1)m - 6$ if

$$k > 4 + \frac{31}{m}.$$

Hence, if $k > 4 + \frac{31}{m}$, then we should stop rolling if our score, n, is in the interval

$$km < n < (k+1)m$$

and possibly earlier.

Suppose $m = 10$. Then $4 + \frac{31}{10} = 7.1$, so (with $k = 8 > 7.1$) we should definitely stop rolling if our score is between 84 and 89.

We can then work out an optimal strategy recursively as follows.

Let $E(x)$ be the expected value of our ultimate score if our sum is ever x.

Then $E(84) = 84$, $E(85) = 85$, $E(86) = 86$, $E(87) = 87$, $E(88) = 88$, and $E(89) = 89$ (since we will stop at any of those scores). We also know $E(x) = 0$ if x is a positive multiple of m.

We can then define $f(n) = \dfrac{1}{6}\displaystyle\sum_{i=1}^{6} E(n+i)$. Then, if $f(n) > n$, we should roll when our sum is n and $E(n) = f(n)$. On the other hand, if $f(n) \leq n$, we should stop rolling when our sum is n and $E(n) = n$.

In this way, we can calculate $E(n)$ downward from $n = 83$ to $n = 0$, noting whether we stop rolling or not to create our optimal strategy and the expected value of our strategy.

Writing a little code, we can thus find that we should roll unless the sum is 24 or 25 or greater than 33. With this strategy, the expected score is

$$\frac{16233112301105386288443 1}{1228188442892963099443 2} = 13.21711859042473....$$

Applying this same method to other values of m, we have the following table.

m	stop if	expected value
6	$n \geq 19$	7.221451623108286812
7	$n \geq 15$	8.585032818442838864
8	$n = 18$ or $n \geq 26$	10.12982919499704393
9	$n = 21, 22$ or $n \geq 30$	11.67916132417996147
10	$n = 24, 25,$ or $n \geq 33$	13.21711859042473150
11	$n = 27, 28$ or $n \geq 38$	14.72823564563309959
12	$n = 30, 31$ or $n \geq 42$	16.27534383168068736
13	$n = 33, 35, 35$ or $n \geq 46$	17.90549414976900364
14	$n = 36, 37, 38$ or $n \geq 50$	19.43362157318550401
15	$n = 39, 40, 41$ or $n \geq 54$	20.97094434047380285
16	$n = 42, 43, 44, 58, 59, 60, 61, 62$ or $n \geq 74$	22.51571524339529867
17	$n = 45, 46, 47, 62, 63, 64, 65, 66$ or $n \geq 79$	24.07230800164414883
18	$n = 48, 49, 50, 66, 67, 68, 69, 70$ or $n \geq 84$	25.64211352850069779
19	$n = 51, 52, 53, 70, 71, 72, 73, 74$ or $n \geq 89$	27.21360753502956739
20	$n = 54, 55, 56, 74, 75, 76, 78$ or $n \geq 94$	28.75912955252540060

For $2 \leq m \leq 5$, it is probably best to attack each one separately, which perhaps I will do some other time.

55. *Suppose we play a game with a die in which we use two rolls of the die to create a two digit number. The player rolls the die once and decides which of the two digits they want that roll to represent. Then, the player rolls a second time and this determines the other digit. For instance, the player might roll a 5, and decide this should be the "tens" digit, and then roll a 6, so their resulting number is 56.*

    What strategy should be used to create the largest number on average? What about the three digit version of the game?

    A strategy in this game is merely a rule for deciding whether the first roll should be the "tens" digit or the "ones" digit. If the first roll is a 6, then it must go in the "tens" digit, and if it's a 1, then it must go in the "ones" digit. This leaves us with what to do with 2,3,4 and 5. If the first roll is b, then using it as the "ones" digit results in an expected number of $\frac{7}{2} \cdot 10 + b$. Using it as the "tens" digit results in an expected number of $10b + \frac{7}{2}$. So, when is $10b + \frac{7}{2} > \frac{7}{2} \cdot 10 + b$? When $b \geq 4$. Thus, if the first roll is 4, 5 or 6, the player should use it for the "tens" digit. With this strategy, the expected value of the number is

$$\frac{1}{6}(63.5 + 53.5 + 43.5 + 38 + 37 + 36) = 45.25.$$

    In the three-digit version of the game, once we have decided what to do with the first roll, we'll be done, since we will then be in the two-digit case which we solved above. Note this is obviously true if we place the first roll in the "hundreds" digit. If we place the first roll in the "ones" digit, then the strategy to maximize the resulting number is the same as the two-digit case, simply multiplied by a factor of ten. If we place the first roll in the "tens" digit, then our strategy is to put the next roll b in the "hundreds" digit if

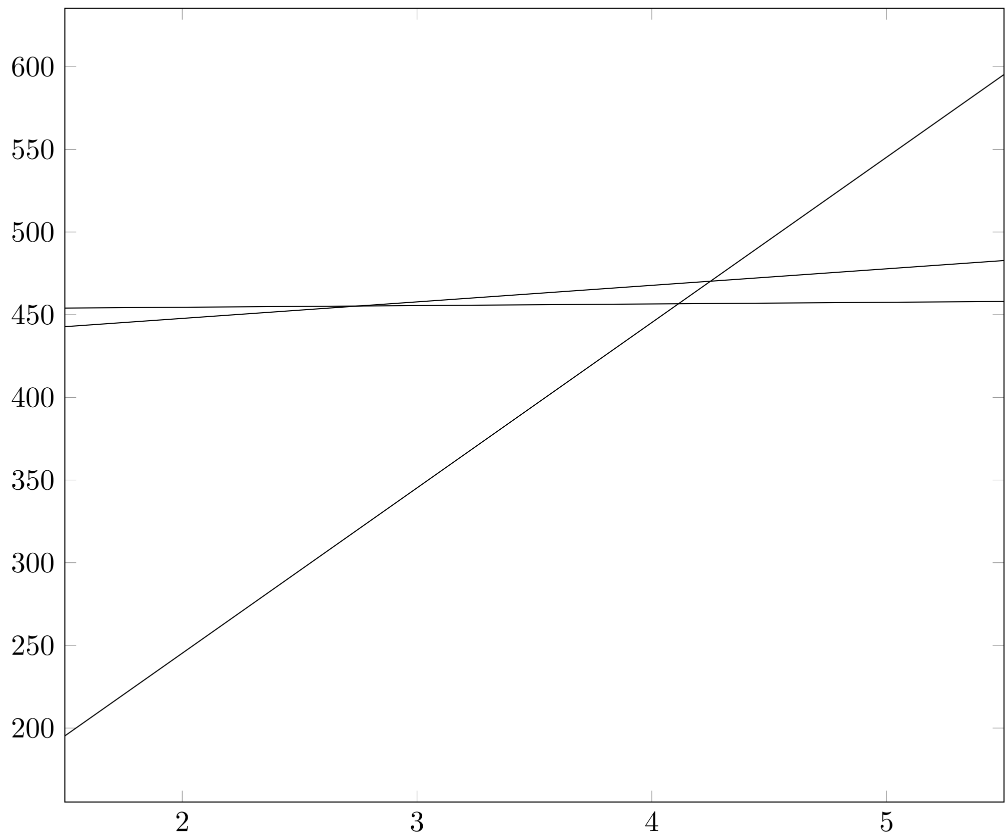
$$100b + 3.5 > 350 + b$$

i.e., if $b \geq 4$. Thus we have the same strategy in all three cases: put the second roll in the largest digit if it is at least 4.

Now, if the first roll, b, is placed in the "hundreds" digit, then the expected value will be $100b + 45.25$. If the first roll is placed in the "ones" digit, then the expected value will be $452.5 + b$. If the first roll is places in the "tens" digit, then the expected value will be

$$10b + (351 + 352 + 353 + 403.5 + 503.5 + 603.5)/6 = 427.75 + 10b.$$

Our strategy thus comes down to maximizing the quantities $100b + 45.25$, $427.75 + 10b$, and $452.5 + b$. From the graph below, we see that $100b + 45.25$ is the largest when $b \geq 5$; $427.75 + 10b$ is largest when $3 \leq b \leq 4$, and $452.5 + b$ is largest when $b < 3$. Thus our strategy for the first roll is this: if it is at least 5, put it in the "hundreds" digit; if it is 3 or 4, put it in the "tens" digit; otherwise, put it in the ones digit. If the second roll is 4, 5, or 6, place it in the largest available digit.



The expected value using this strategy is thus

$$(645.25 + 545.25 + (40 + 427.75) + (30 + 427.75) + (452.5 + 2) + (452.5 + 1))/6 = 504.$$

56. *Suppose we roll a single die, repeatedly if we like, and sum. We can stop at any point, and the sum becomes our score; however, if we exceeed* 10*, our score is zero.*

    What should our strategy be to maximize the expected value of our score? What is the expected score with this optimal strategy?

    What about limits besides 10*?*

    We consider the game first with a limit of 10.

We need to decide, if our current score is n, whether or not we should continue rolling.

Suppose our current score is 10. Then rolling will not help, since our score would become zero. So we must "stick" (i.e., not roll) if our score is 10.

Suppose our current score is 9. Then rolling will give us an expected score of

$$\left(\frac{1}{6}\right)10 + \left(\frac{5}{6}\right)0 = \frac{10}{6} < 9.$$

Since the expected value is less than 9, it is better to stick on 9 than to roll.

Suppose our current score is 8. Then rolling will give us an expected score of

$$\left(\frac{1}{6}\right)9 + \left(\frac{1}{6}\right)10 + \left(\frac{4}{6}\right)0 = \frac{19}{6} < 8$$

so we should stick on 8.

Suppose our current score is 7. Then rolling will give us an expected score of

$$\left(\frac{1}{6}\right)8 + \left(\frac{1}{6}\right)9 + \left(\frac{1}{6}\right)10 + \left(\frac{3}{6}\right)0 = \frac{9}{2} < 7$$

so we should stick on 7.

Suppose our current score is 6. Then rolling will give us an expected score of

$$\left(\frac{1}{6}\right)7 + \left(\frac{1}{6}\right)8 + \left(\frac{1}{6}\right)9 + \left(\frac{1}{6}\right)10 + \left(\frac{2}{6}\right)0 = \frac{17}{3} < 6$$

so we should stick on 6.

Suppose our current score is 5. Then rolling will give us an expected score of

$$\left(\frac{1}{6}\right)6 + \left(\frac{1}{6}\right)7 + \left(\frac{1}{6}\right)8 + \left(\frac{1}{6}\right)9 + \left(\frac{1}{6}\right)10 + \left(\frac{1}{6}\right)0 = \frac{20}{3} > 5.$$

Since the expected value is greater than 5, we should roll, even though there is a chance that we will end up with a score of zero.

If our current score is less than 4, then there is no chance that one more roll will result in a score of zero (i.e., a lower score than the current score), so it is always better to roll.

Hence, the optimal strategy is: roll again if the score is 5 or less, and stick otherwise.

To calculate the expected final score with this strategy, let $E(m)$ be the expected final score starting with a current score of m. Then we seek $E(0)$.

If $m > 5$, $E(m) = m$.

If $m = 5$, we have $E(5) = \frac{1}{6}\sum_{i=6}^{10} E(i) = \frac{20}{3}$.

Then

$$E(4) = \frac{1}{6}\sum_{i=1}^{6} E(4+i) = \frac{70}{9}$$

$$E(3) = \frac{1}{6}\sum_{i=1}^{6} E(3+i) = \frac{200}{27}$$

$$E(2) = \frac{1}{6}\sum_{i=1}^{6} E(2+i) = \frac{1157}{162}$$

$$E(1) = \frac{1}{6}\sum_{i=1}^{6} E(1+i) = \frac{6803}{972}$$

$$E(0) = \frac{1}{6}\sum_{i=1}^{6} E(i) = \frac{40817}{5832}$$

Thus, the expected value of the score is $\frac{40817}{5832} = 6.99879972565157...$, just the tiniest bit less than 7.

What about other limits?

Suppose the limit is k (so we lose all points if the score ever exceeeds k).

Then if our current score is $n = k-1$, our expected score if we roll is $\frac{1}{6}(n+1)$ which is greater than n if and only if $n < \frac{1}{5}$, i.e., $n = 0$. Thus, should always stop on if our current score is $k-1$ or greater.

Suppose our current score is $n = k-2$. If we roll, our expected score is $\frac{1}{6}(n+1+n+2) = \frac{1}{6}(2n+3)$ which is greater than n if and only if $n < \frac{3}{4}$, i.e., $n = 0$. So we should always stop if our current score is $k-2$ or greater.

Suppose our current score is $n = k-3$. Then the expected score if we roll is $\frac{1}{6}(n+1+n+2+n+3) = \frac{1}{6}(3n+6)$ which is greater than n if and only if $n < 2$, i.e., $k < 5$. Hence, we should stick on $k-3$ or greater, unless $k = 4$, in which case we should roll on $k-3$.

Suppose our current score is $n = k-4$. Then the expected score if we roll is $\frac{1}{6}(n+1+n+2+n+3+n+4) = \frac{1}{6}(4n+10)$ which is greater than n if and only if $n < 5$, i.e., $k < 9$. Hence we should stick on $k-4$ or greater, unless $k = 5, 6, 7$, or 8, in which case we should roll on $k-4$.

Suppose our current score is $n = k-5$. Then the expected score if we roll is $\frac{1}{6}(n+1+n+2+n+3+n+4+n+5) = \frac{1}{6}(5n+15)$ which is greater than n if and only if $n < 15$, i.e., $k < 20$. Hence, we should stick on $k-5$ or greater if $k \geq 20$, and otherwise roll on $k-5$.

Let k be the limit in our game, and let s be the *sticking value*, i.e., we should stop rolling if out score is at or above s. Then, for all $k \geq 20$, the sticking value is $k-5$. For smaller values of k, the s values are summarized in the table below.

What are the expected scores using this optimal strategy?

For any k, let $E(x)$ be the expected value of our final score if our score is x.

If $k = 1$, then the expected score is $\frac{1}{6}(1) = \frac{1}{6}$.

If $k = 2$, then the expected score is $\frac{1}{6}(1) + \frac{1}{6}(2) = \frac{1}{2}$.

If $k = 3$, then the expected score is $\frac{1}{6}(1+2+3) = 1$.

If $k = 4$, then the expected score is $\frac{1}{6}(2+3+4) + \frac{1}{6}(\frac{1}{6}(2+3+4)) = \frac{7}{4} = 1.75$.

If $k = 5$, then the expected score is $\frac{1}{6}(2+3+4+5) + \frac{1}{6}(\frac{1}{6}(2+3+4+5)) = \frac{49}{18} = 2.7\bar{2}$.

Suppose $k = 6$. We stick on 3 or greater. Then $E(3) = 3, E(4) = 4, E(5) = 5$, and $E(6) = 6$. Then $E(2) = \frac{1}{6}(E(3)+E(4)+E(5)+E(6))$ and $E(1) = \frac{1}{6}(E(2)+E(3)+E(4)+E(5)+E(6))$. Finally, the expected score is

$$\frac{1}{6}\sum_{i=1}^{6} E(i) = \frac{49}{12} = 4.08\bar{3}.$$

Suppose $k = 7$. We stick on 4 or greater. Then $E(4) = 4, E(5) = 5, E(6) = 6$, and $E(7) = 7$, and we may set $E(x) = 0$ if $x > 7$. Then $E(j) = \frac{1}{6}\sum_{i=1}^{6} E(j+i)$ for $j = 0, 1$ and 2. The expected score is then $E(0) = \frac{3017}{648} \approx 4.6558642$.

Using the same technique, we have the following table of results.

k	s	$E(0)$	$E(0)$ approx.
1	1	1/6	0.166666
2	1	1/2	0.5
3	1	1	1
4	2	7/4	1.75
5	2	49/18	2.722222
6	3	49/12	4.083333
7	4	3017/648	4.655864
8	5	20629/3888	5.305813
9	5	46991/7776	6.043081
10	6	40817/5832	6.998800
11	7	84035/10368	8.105228
12	8	7695859/839808	9.163831
13	9	101039827/10077696	10.026084
14	10	55084715/5038848	10.932006
15	11	4307852885/362797056	11.874002
16	12	13976066993/1088391168	12.841033
17	13	60156375517/4353564672	13.817729
18	14	1131329591/76527504	14.783307
19	15	7388908661401/470184984576	15.714897
20	15	2611098542647/156728328192	16.660029
50	45		46.666667
100	95		96.666667

The calculations suggest that the expected score approaches $k - \frac{10}{3}$ as k goes to infinity. Proving that would be nice.

Let $f(k, m)$ be the expected score of the game with a limit of k where we use the strategy of stopping on m or greater. We can calculate $f(k, m)$ in PARI/GP with the following code:

```
f(k, m) = A=vector(k+6);for(i=m,k,A[i]=i);
for(j=1,m-1,A[m-j]=1/6*sum(n=1,6,A[m-j+n])); return(1/6*sum(i=1,6,A[i]))
```

Chapter 4

Problems for the future

Here are some problems that I intend to add to this collection some time in the future, as soon as I get around to writing decent solutions.

1. More Drop Dead: probability of getting zero? probability of any particular value?

2. More Threes: probability of getting any particular score?

3. Law of Large Numbers related: What is the expected number of rolls of a single die needed until there is a 99% chance that the proportion of 2s (say) thrown is within some specified interval around 1/6 (e.g., $1/6 - 0.01 \leq r \leq 1/6 + 0.01$?

4. If a die is rolled 100 times (say), what is the probability that all six sides have appeared at least once? If a die is rolled 100 times (say), what is the probability that there is a run of 6 that has all six possible faces? What about a run of 10 with all six faces at least once?

5. For every composite n, there appear to pairs of "weird" dice with n sides (i.e. a pair of dice not numbered in the usual way with sum probabilities equal to the standard dice). Prove this. For many n, there are many such pairs. Give useful bounds on the number of such pairs in terms of n.

   For $n = 4k + 2$, it appears that the dice

   $$\{1, 2, 2, 3, 3, ..., 2k{+}3, 2k{+}3, 2k{+}4\}, \{1, 3, 5, ..., 2k{+}1, 2k{+}2, 2k{+}3, ..., n{-}1, n, n{+}2, n{+}4, ..., 6k{+}2\}$$

   do the trick.

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Appendix A

Dice sum probabilities

Sums of 2, 6-Sided Dice

Sum	Probability
2	1/36
3	2/36 = 1/18
4	3/36 = 1/12
5	4/36 = 1/9
6	5/36
7	6/36 = 1/6
8	5/36
9	4/36 =1/9
10	3/36 =1/12
11	2/36 =1/18
12	1/36

Sums of 3, 6-Sided Dice

Sum	Probability
3	1/216
4	3/216 = 1/72
5	6/216 = 1/36
6	10/216 = 5/108
7	15/216 = 5/72
8	21/216 = 7/72
9	25/216
10	27/216 = 1/8
11	27/216 = 1/8
12	25/216
13	21/216 = 7/72
14	15/216 = 5/72
15	10/216 = 5/108
16	6/216 = 1/36
17	3/216 = 1/72
18	1/216 = 1/216

Appendix B

Handy Series Formulas

For $|r| < 1$,

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \tag{B.1}$$

For $|r| < 1$,

$$\sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r} \tag{B.2}$$

For $|r| < 1$,

$$\sum_{n=1}^{\infty} nr^n = \frac{r}{(1-r)^2} \tag{B.3}$$

For $|r| < 1$,

$$\sum_{n=1}^{N} nr^n = \frac{r\left(1 - r^N\ (1 + N - N\,r)\right)}{(1-r)^2} \tag{B.4}$$

For $|r| < 1$,

$$\sum_{n=1}^{\infty} (n+k)r^n = \frac{r(1+k-kr)}{(1-r)^2} \tag{B.5}$$

For $|r| < 1$,

$$\sum_{n=1}^{\infty} n^2 r^n = \frac{r(1+r)}{(1-r)^3} \tag{B.6}$$

Appendix C

Dice Sums and Generating Functions

Very often in mathematics a good choice of notation can take you a long way. An example of this is the following method for representing sums of dice. Suppose we have an n-sided die, with sides $1, 2, ..., n$ that appear with probability $p_1, p_2, ..., p_n$, respectively. Then, if we roll the die twice and add the two rolls, the probability that the sum is k is given by

$$\sum_{j=1}^{n} p_j p_{k-j} = \sum_{j=k-1}^{n-k} p_j p_{k-j} \tag{C.1}$$

if we say $p_i = 0$ if $i < 1$ or $i > n$.

Now consider the following polynomial:

$$P = p_1 x + p_2 x^2 + \cdots + p_n x^n \tag{C.2}$$

If we square P, we get

$$P^2 = a_2 x^2 + a_3 x^3 + \cdots + a_{2n} x^{2n} \tag{C.3}$$

where a_k, for k=2, 3,…, $2n$, is given by

$$a_k = \sum_{j=k-1}^{n-k} p_j p_{k-j}. \tag{C.4}$$

In other words, the probability of rolling the sum of k is the same as the coefficient of x^k in the polynomial given by squaring the polynomial P.

Here's an example. Suppose we consider a standard 6-sided die. Then

$$P = \frac{1}{6}x + \frac{1}{6}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{6}x^5 + \frac{1}{6}x^6 \tag{C.5}$$

and so

$$\begin{aligned} P^2 &= \frac{x^2}{36} + \frac{2x^3}{36} + \frac{3x^4}{36} + \frac{4x^5}{36} + \frac{5\,x^6}{36} + \frac{6x^7}{36} + \frac{5\,x^8}{36} + \frac{4x^9}{36} + \frac{3x^{10}}{36} + \frac{2x^{11}}{36} + \frac{x^{12}}{36} && \text{(C.6)} \\ &= \frac{x^2}{36} + \frac{x^3}{18} + \frac{x^4}{12} + \frac{x^5}{9} + \frac{5\,x^6}{36} + \frac{x^7}{6} + \frac{5\,x^8}{36} + \frac{x^9}{9} + \frac{x^{10}}{12} + \frac{x^{11}}{18} + \frac{x^{12}}{36} && \text{(C.7)} \end{aligned}$$

And so, we see that the probability of rolling a sum of 9, for instance, is $1/9$.

For two different dice the method is the same. For instance, if we roll a 4-sided die, and a 6-sided die, and sum them, the probability that the sum is equal to k is give by the coefficient on x^k in the polynomial

$$\left(\frac{x}{4} + \frac{x^2}{4} + \frac{x^3}{4} + \frac{x^4}{4}\right)\left(\frac{x}{6} + \frac{x^2}{6} + \frac{x^3}{6} + \frac{x^4}{6} + \frac{x^5}{6} + \frac{x^6}{6}\right) \tag{C.8}$$

which, when expanded, is

$$\frac{x^2}{24} + \frac{x^3}{12} + \frac{x^4}{8} + \frac{x^5}{6} + \frac{x^6}{6} + \frac{x^7}{6} + \frac{x^8}{8} + \frac{x^9}{12} + \frac{x^{10}}{24}. \tag{C.9}$$

Notice that this can be written as

$$\frac{1}{24}\left(x^2 + 2\,x^3 + 3\,x^4 + 4\,x^5 + 4\,x^6 + 4\,x^7 + 3\,x^8 + 2\,x^9 + x^{10}\right) \tag{C.10}$$

In general, a fair n-sided die can be represented by the polynomial

$$\frac{1}{n}\left(x + x^2 + x^3 + \cdots + x^n\right) \tag{C.11}$$

With this notation, many questions about dice sums can be transformed into equivalent questions about polynomials. For instance, asking whether or not there exist other pairs of dice that give the same sum probabilities as a pair of standard dice is the same as asking: in what ways can the polynomial

$$(x + x^2 + x^3 + x^4 + ... + x^n)^2$$

be factored into two polynomials (with certain conditions on the degrees and coefficients of those polynomials)?

Using more general terminology, such polynomials are called *generating functions*. They can be applied in lots of situations involving discrete random variables, including sitations in which the variables take on infinitely many values: in such cases, the generating function will be a power series.

Appendix D

Markov Chain Facts

A *Markov chain* is a mathematical model for describing a process that moves in a sequence of steps through a set of states. A finite Markov chain has a finite number of states, $\{s_1, s_2, \ldots, s_n\}$. When the process is in state s_i, there is a probability p_{ij} that the process will next be in state s_j. The matrix $P = (p_{ij})$ is called the *transition matrix* for the Markov chain. Note that the rows of the matrix sum to 1.

The ij-th entry of P^k (i.e. the k-th power of the matrix P) gives the probability of the process moving from state i to state j in exactly k steps.

An *absorbing state* is one which the process can never leave once it is entered. An *absorbing chain* is a chain which has at least one absorbing state, and starting in any state of the chain, it is possible to move to an absorbing state. In an absorbing chain, the process will eventually end up in an absorbing state.

Let P be the transition matrix of an absorbing chain. By renumbering the states, we can always rearrange P into *canonical form*:

$$P = \left(\begin{array}{c|c} Q & R \\ \hline O & J \end{array}\right)$$

where J is an identity matrix (with 1's on the diagonal and 0's elsewhere) and O is a matrix of all zeros. Q and R are non-negative matrices that arise from the transition probabilities between non-absorbing states.

The series $N = I + Q + Q^2 + Q^3 + \ldots$ converges, and $N = (I - Q)^{-1}$. The matrix N gives us important information about the chain, as the following theorem shows.

Theorem 1 *Let P be the transition matrix for an absorbing chain in canonical form. Let $N = (I - Q)^{-1}$. Then:*

- *The ij-th entry of N is the expected number of times that the chain will be in state j after starting in state i.*
- *The sum of the i-th row of N gives the mean number of steps until absorbtion when the chain is started in state i.*
- *The ij-th entry of the matrix $B = NR$ is the probability that, after starting in non-absorbing state i, the process will end up in absorbing state j.*

An *ergodic* chain is one in which it is possible to move from any state to any other state (though not necessarily in a single step).

A *regular* chain is one for which some power of its transition matrix has no zero entries. A regular chain is therefore ergodic, though not all ergodic chains are regular.

Theorem 2 *Suppose P is the transition matrix of an ergodic chain. Then there exists a matrix A such that*

$$\lim_{k\to\infty} \frac{P + P^2 + P^3 + \cdots + P^k}{k} = A$$

For regular chains,

$$\lim_{k\to\infty} P^k = A.$$

The matrix A has each row the same vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. One way to interpret this is to say that the long-term probability of finding the process in state i does not depend on the initial state of the process.

The components $a_1, a_2, \ldots, a_n$ are all positive. The vector **a** is the unique vector such that

$$a_1 + a_2 + \cdots + a_n = 1$$

and

$$\mathbf{a}P = \mathbf{a}$$

For this reason, **a** is sometimes called the *fixed point probability vector*.

The following theorem is sometimes called the *Mean First Passage Theorem*.

Theorem 3 *Suppose we have a regular Markov chain, with transition matrix P. Let $E = (e_{ij})$ be a matrix where, for $i \neq j$, e_{ij} is the expected number of steps before the process enters state j for the first time after starting in state i, and e_{ii} is the expected number of steps before the chain re-enters state i. Then*

$$E = (I - Z + JZ')D$$

where $Z = (I - P - A)^{-1}$, $A = \lim_{k\to\infty} P^k$, Z' is the diagonal matrix whose diagonal entries are the same as Z, J is the matrix of all 1's, and D is a diagonal matrix with $D_{ii} = 1/A_{ii}$.

Appendix E

Linear Recurrence Relations

Here's a useful theorem:

Theorem 1 *Consider the linear recurrence relation*

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_k. \tag{E.1}$$

If the polynomial equation (known as the characteristic equation of the recurrence relation)

$$x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_k = 0 \tag{E.2}$$

has k distinct roots $r_1, ...r_k$, then the recurrence relation E.1 has as a general solution

$$x_n = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n. \tag{E.3}$$

for some constants $c_1, \ldots, c_k$.

Proof: We can prove this with a bit of linear algebra, but we'll do that some other time.
Example: May as well do the old classic. The Fibonacci numbers are defined by

$$f_0 = 1, f_1 = 1, \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n > 1.$$

The characteristic equation is

$$x^2 - x - 1 = 0$$

which has roots

$$r_1 = \frac{1+\sqrt{5}}{2} \text{ and } r_2 = \frac{1-\sqrt{5}}{2}.$$

So

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for constants A and B. Since

$$f_0 = 1 = A + B$$

and

$$f_1 = 1 = A + B + \frac{\sqrt{5}}{2}(A - B)$$

we conclude that

$$A = \frac{1+\sqrt{5}}{2\sqrt{5}} \text{ and } B = \frac{-1+\sqrt{5}}{2\sqrt{5}}$$

so that

$$f_n = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}.$$

Appendix F

Drop Dead Simulation Data

In 10^7 simulated games of Drop Dead, the following frequencies of scores were achieved.

score (n)	number of games with score n	number of games with score $\leq n$
0	2044719	2044719
1	212885	2257604
2	69266	2326870
3	226785	2553655
4	353402	2907057
5	180453	3087510
6	338023	3425533
7	335320	3760853
8	220094	3980947
9	258632	4239579
10	321143	4560722
11	233060	4793782
12	259303	5053085
13	257301	5310386
14	240123	5550509
15	225187	5775696
16	241364	6017060
17	223916	6240976
18	211570	6452546
19	208928	6661474
20	199351	6860825
21	184160	7044985
22	179823	7224808
23	170278	7395086
24	158398	7553484
25	150177	7703661
26	144061	7847722
27	133669	7981391
28	126951	8108342
29	120721	8229063
30	112686	8341749
31	107082	8448831
32	101481	8550312
33	95625	8645937
34	90543	8736480
35	85022	8821502

36	80163	8901665
37	74323	8975988
38	70833	9046821
39	66156	9112977
40	61407	9174384
41	57766	9232150
42	53754	9285904
43	50170	9336074
44	47367	9383441
45	44263	9427704
46	41087	9468791
47	38652	9507443
48	35709	9543152
49	32950	9576102
50	31230	9607332
51	28709	9636041
52	26898	9662939
53	25022	9687961
54	23160	9711121
55	21400	9732521
56	19890	9752411
57	18385	9770796
58	17360	9788156
59	16018	9804174
60	14872	9819046
61	13627	9832673
62	12830	9845503
63	11798	9857301
64	10968	9868269
65	10345	9878614
66	9332	9887946
67	8584	9896530
68	8028	9904558
69	7478	9912036
70	6846	9918882
71	6220	9925102
72	5928	9931030
73	5411	9936441
74	5091	9941532
75	4660	9946192
76	4263	9950455
77	3796	9954251
78	3750	9958001
79	3401	9961402
80	3077	9964479
81	2940	9967419
82	2674	9970093
83	2455	9972548
84	2174	9974722
85	2087	9976809
86	2019	9978828
87	1701	9980529
88	1650	9982179
89	1504	9983683

90	1334	9985017
91	1234	9986251
92	1149	9987400
93	1056	9988456
94	975	9989431
95	845	9990276
96	817	9991093
97	772	9991865
98	676	9992541
99	615	9993156
100	583	9993739
101	539	9994278
102	501	9994779
103	460	9995239
104	395	9995634
105	354	9995988
106	326	9996314
107	318	9996632
108	285	9996917
109	247	9997164
110	223	9997387
111	225	9997612
112	194	9997806
113	160	9997966
114	175	9998141
115	148	9998289
116	140	9998429
117	119	9998548
118	138	9998686
119	105	9998791
120	99	9998890
121	96	9998986
122	105	9999091
123	64	9999155
124	75	9999230
125	54	9999284
126	54	9999338
127	65	9999403
128	52	9999455
129	48	9999503
130	45	9999548
131	28	9999576
132	32	9999608
133	33	9999641
134	37	9999678
135	27	9999705
136	27	9999732
137	34	9999766
138	12	9999778
139	23	9999801
140	21	9999822
141	12	9999834
142	17	9999851
143	12	9999863

144	13	9999876
145	14	9999890
146	15	9999905
147	5	9999910
148	13	9999923
149	6	9999929
150	5	9999934
151	7	9999941
152	5	9999946
153	8	9999954
154	3	9999957
155	2	9999959
156	7	9999966
157	1	9999967
158	2	9999969
159	1	9999970
160	2	9999972
161	1	9999973
162	2	9999975
163	3	9999978
164	3	9999981
165	2	9999983
166	4	9999987
167	2	9999989
170	2	9999991
172	1	9999992
174	2	9999994
175	1	9999995
177	2	9999997
178	1	9999998
181	1	9999999
192	1	1000000

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